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Asymptotic Formulas for the Eigenvalues of the Timoshenko Beam[1]

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Asymptotic formulas are derived for the eigenvalues of a free-ended Timoshenko beam which has variable mass density and constant beam parameters otherwise. These asymptotic formulas show how the eigenvalues (and hence how the natural frequencies) of such a beam depend on the material and geometric parameters which appear as coefficients in the Timoshenko differential equations.

Key Words: eigenvalue asymptotics; Timoshenko beam models.

1. INTRODUCTION

Suppose a structural beam is driven by a laterally oscillating sinusoidal force. As the frequency of this applied force is varied, the response varies. Experimental frequencies for which the response is maximized are called natural frequencies of the beam. Our goal is to address the question: If a beam's natural frequencies are known, what can be inferred about its

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bending stiffnesses or its mass density? To answer this question we need to know asymptotic formulas for these frequencies. Here we establish such formulas for beams with variable mass density but otherwise constant beam parameters. We make this assumption as a first step toward solving the problem with both variable density and variable stiffness. We also make this assumption because it is consistent with some applications of interest to us. An example of an application consistent with our assumption is an aircraft wing with struts which have been added so that there is an appreciable change in the density and a minimal change in stiffness.

One widely used mathematical model for describing the transverse vibration of beams was developed by Stephen Timoshenko in the 1920s. This model is chosen because it is a more accurate model than the Euler–Bernouli beam model and because systems of Timoshenko beam models are used to model aircraft wings. The mathematical equations that arise are two coupled partial differential equations,

$$(EI\psi_x)_x + kAG(w_x - \psi) - \rho I\psi_{tt} = 0,$$

$$\big(kAG(w_x - \psi)\big)_x - \rho A w_{tt} = P(x,t).$$

The dependent variable $w = w(x,t)$ represents the lateral displacement at time t of a cross section located x units from one end of the beam. $\psi = \psi(x,t)$ is the cross sectional rotation due to bending. E is Young's modulus, i.e., the modulus of elasticity in tension and compression, and G is the modulus of elasticity in shear. The non-uniform distribution of shear stress over a cross section depends on cross sectional shape. The coefficient k is introduced to account for this geometry dependent distribution of shearing stress. I and A represent cross sectional inertia and area, ρ is the mass density of the beam per unit length, and $P(x,t)$ is an applied force. If we suppose the beam is anchored so that the so-called "free-free" boundary conditions hold (i.e., shearing forces and moments are assumed to be zero at each end of the beam), then w and ψ must satisfy the following four boundary conditions,

$$w_x - \psi|_{x=0,L} = 0, \qquad \psi_x|_{x=0,L} = 0. \tag{1}$$

After making a standard separation of variables argument, one finds that the Timoshenko differential equations for w and ψ lead to a coupled system of two second order ordinary differential equations for $y(x)$ and $\psi(x)$,

$$(EI\Psi_x)_x + kAG(y_x - \Psi) + p^2 I\rho\Psi = 0, \tag{2}$$

$$\big(kAG(y_x - \Psi)\big)_x + p^2 A\rho y = 0. \tag{3}$$

Here, p^2 is an eigenvalue parameter. The conditions on w and ψ in (1) imply y and Ψ must satisfy the same free-free boundary conditions. We must have

$$y_x - \Psi|_{x=0,L} = 0, \qquad \Psi_x|_{x=0,L} = 0. \tag{4}$$

This boundary value problem for y and Ψ is self-adjoint. Hence, the values of p^2 for which nontrivial solutions to this problem exist, i.e., the eigenvalues for this model, are real. Furthermore, it is not difficult to show that the collection of all eigenvalues for this problem forms a discrete, countable, unbounded set of real non-negative numbers. Moreover, it can be shown that if σ is a natural frequency for a beam, then $p^2 = (2\pi\sigma)^2$ is one of the beam's eigenvalues. Therefore, it is possible to determine eigenvalues from natural frequency data obtained in an experiment like the one indicated in the opening paragraph.

Suppose from vibration experiments we have determined a set of natural frequencies for a beam with unknown elastic moduli and mass density and have constructed a sequence of eigenvalues from this data. What information can the eigenvalues provide about these unknown material parameters? To address this question, we must determine how eigenvalues depend on E, I, kG, A, and ρ. This determination is not easy, since the dependence of eigenvalues on these coefficients is highly nonlinear. Another difficulty arises because the Timoshenko boundary value problem involves two second order differential equations. When the coefficients in these differential equations are non-constant, the system of two second order equations cannot be transformed into a single fourth order equation. Therefore, to make progress in the case where coefficients are non-constant, the boundary-value problem must be handled as a system of equations.

For a simpler, Sturm–Liouville type boundary value problem,

$$\begin{gathered} y''(z) + (\lambda - q(z))y(z) = 0, \qquad 0 \le z \le 1, \\ y'(0) - hy(0) = y'(1) + Hy(1) = 0, \end{gathered} \tag{5}$$

it is known that for square integrable $q(z)$, nontrivial solutions $y(z)$ for this problem exist if and only if $\lambda = \mu_n$, where

$$\begin{gathered} \mu_n = n^2\pi^2 + C_q - \int_0^1 q(z)\cos(2n\pi z)\,dz + \alpha_n, \\ C_q = 2h + 2H + \int_0^1 q(z)\,dz, \ \sum_{n=1}^{\infty} \alpha_n^2 < \infty. \end{gathered} \tag{6}$$

(See Hald and McLaughlin [12, pp. 313–314] as well as Isaacson and Trubowitz [14], Borg [2], and Fulton and Pruess [6, 7]; for other

Sturm–Liouville equations with, effectively, less smooth coefficients, see Coleman and McLaughlin [5].) The importance of Eq. (6) is that it shows how the eigenvalues for the Sturm–Liouville problem are related to the coefficient $q(z)$ appearing in (5). Algorithms for reconstructing q from spectral data rely strongly on asymptotic formulas like the one given in (6). (For example, see Hald [11] and Rundell and Sacks [18].)

Returning now to the Timoshenko beam equations, we ask the question: What information is contained in the eigenvalues for the Timoshenko beam? Given a sequence of eigenvalues, can we infer knowledge about the beam parameters which give rise to these eigenvalues? Asymptotic formulas for the Sturm–Liouville eigenvalues are critical to determining q from spectral data. We expect that analogous formulas for the Timoshenko eigenvalues will play a key role in recovering beam parameters like E, kG, or ρ from such data. In this paper, our objective is to determine asymptotic formulas for the eigenvalues of the Timoshenko beam when free-free boundary conditions are enforced and when ρ is allowed to vary. We suppose that E, kG, A, and I are constants and assume ρ is a positive function of x on $[0, L]$ such that $0 = \rho_x(0) = \rho_x(L)$ and ρ_{xx} is in $L_\infty(0, L)$. Under these assumptions, we derive asymptotic formulas for the eigenvalues of the free-free Timoshenko beam.

In the next three sections, approximations are derived (accurate to within an error that is $O(1/p)$) for the square roots of eigenvalues of free-free beams with variable density. An important step in deriving these preliminary approximations of the eigenvalues is the use of a transformation (see Section 1) which changes the Timoshenko differential equations (2) and (3) into a new pair of differential equations, where in the new equations, the coefficient which contains the eigenvalue parameter p^2 no longer depends on the independent variable. The key feature of the transformed system is that the largest terms in the new differential equations, and hence the most important terms, are multiplied by coefficients which do not depend on the new independent variable. As the eigenvalue parameter grows, solutions to the transformed differential equations approach the solutions of a certain set of constant coefficient differential equations. It is therefore possible to derive an approximate solution (accurate to within an $O(1/p)$ error) to an initial value problem in which initial conditions are chosen so that the left transformed boundary conditions are enforced. In Section 2, this initial value problem is presented and its approximate solution is derived. By applying the two remaining transformed right boundary conditions to the approximate solution of the initial value problem, a frequency equation is determined. In Section 3, estimates of square-roots of eigenvalues are made from this frequency equation. These estimates appear in Theorem 4.3.

Our approach to deriving the final asymptotic formulas (Section 4) is built from the following idea. Suppose $\rho_0 \equiv (1/\mathrm{L} \int_0^L \rho^{1/2}(x)\,dx)^2$, and let $\hat{\rho}(x;t) \equiv \rho_0 + t\tilde{\rho}(x)$, where $\tilde{\rho} = \rho - \rho_0$ and t is an auxiliary parameter which we allow to vary from 0 to 1. In the Timoshenko differential equations, let $\rho(x)$ be replaced by $\hat{\rho}(x;t)$. Define $\acute{p}^2$ to be an eigenvalue for a free-free beam with mass density $\hat{\rho}$ and constant material and geometric parameters otherwise. When $t = 0$, $\hat{\rho} = \rho_0$ and $\acute{p}^2$ is an eigenvalue for a beam where E, I, kG, A, and $\hat{\rho} = \rho_0$ are all independent of x. As t increases to 1, $\hat{\rho}$ goes to $\rho(x)$, and $\acute{p}^2$ changes continuously in t to an eigenvalue for a beam with variable density $\rho(x)$. Let $\hat{L}_1 = \int_0^L \hat{\rho}^{1/2}(x;t)\,dx$ and define $\mu^2 \equiv \hat{L}_1^2 \acute{p}^2$. We show that there is a function G such that

$$\frac{d(\mu^2)}{dt} = G(\hat{Y}, \hat{\Phi}), \tag{7}$$

where $(\hat{Y}, \hat{\Phi})$ is a transformed eigenfunction pair corresponding to the eigenvalue $\acute{p}^2$ of a free-free beam with material parameters E, I, kG, A, and $\hat{\rho}$. Integrating (7) formally with respect to t from 0 to 1, we find that

$$\acute{p}^2|_{t=1} - \acute{p}^2|_{t=0} = \frac{1}{L_1}\int_0^1 G(\hat{Y}, \hat{\Phi})\,dt. \tag{8}$$

The term $\acute{p}^2|_{t=0}$ is an eigenvalue for a beam where E, I, kG, A, and $\rho = \rho_0$ are independent of x; i.e., $\acute{p}^2|_{t=0}$ represents an eigenvalue for a uniform beam. Asymptotic formulas for the eigenvalues of free-free and clamped-clamped uniform beams are derived in Geist [8] and published in [10]. From these uniform beam formulas, an asymptotic approximation for the term $\acute{p}^2|_{t=0}$ can be made. The final asymptotic eigenvalue formulas for beams with variable density $\rho(x)$ are obtained by replacing the term $(1/L_1)\int_0^1 G(\hat{Y}, \hat{\Phi})\,dt$ with an asymptotic approximation derived below, and by replacing $\acute{p}^2|_{t=0}$ with the appropriate uniform beam eigenvalue formulas given in [10].

The function G depends only on E, I, kG, A, ρ, and the transformed eigenfunctions $\hat{\Phi}$ and $\hat{Y}$. Approximations to the square roots of eigenvalues given in Theorem 4.3 allows us to determine $\hat{\Phi}$ and $\hat{Y}$ and hence G to within an error that is $O(1/p)$. Then (8) is used to sharpen our estimates of the eigenvalues for the Timoshenko beam. From (8) we compute the final asymptotic formulas, which are given in Theorem 5.2. Note that the advantage of this method over say a variational method is that we can determine more than the first term in the eigenvalue expansion and prove a bound in the remainder no matter how large the difference is between ρ and ρ_0.

2. THE TRANSFORMED PROBLEM

To begin, the free-free Timoshenko boundary value problem is proved equivalent to a certain transformed boundary value problem derived below. This equivalency holds when ρ depends on x; all other beam parameters are assumed constant. A key feature of the transformed problem is that the coefficient containing the eigenvalue parameter no longer depends on the independent variable.

To derive this equivalent problem, a lemma is proved which applies to single second order equations. We will use this lemma to prove Theorem 2.1, in which the Timoshenko system of differential equations is transformed to a new pair of equations.

LEMMA 2.1. *Suppose $\rho(x)$ is positive for all $x \in [0, L]$ and $\rho_{xx}(x) \in L_2[0, L]$. Let $L_1 = \int_0^L \rho^{1/2}(x_1)\,dx_1$ and $z(x) = (1/L_1)\int_0^x \rho^{1/2}(x_2)\,dx_2$. Let β, α, I, and p^2 be constants. Then $v(x) = \rho^{-1/4}V(z(x))$ satisfies the equation*

$$\alpha v'' + \left(p^2\tau\rho - \beta\right)v = f \tag{9}$$

if and only if $V(z)$ satisfies

$$V_{zz} + \left[\frac{(\rho^{-1/4})_{xx}L_1^2}{\rho^{3/4}} + \frac{\tau L_1^2 p^2}{\alpha} - \frac{\beta L_1^2}{\alpha\rho}\right]V = \frac{fL_1^2}{\alpha\rho^{3/4}}. \tag{10}$$

Proof. Suppose $v = A(x)V(z(x))$, where $A(x)$ and $z(x)$ are as yet unspecified smooth functions of x. Then

$$\alpha v'' = \alpha A(z')^2 V_{zz} + \frac{(\alpha A^2 z')'}{A}V_z + \alpha A'' V,$$

and provided $\alpha(z')^2 A \neq 0$ for $x \in [0, L]$, it follows that $\alpha v'' + (p^2\tau\rho - \beta)v = f(x)$ if and only if

$$V_{zz} + \frac{(A^2 z')'}{A^2(z')^2}V_z + \left[\frac{A''}{A(z')^2} + \frac{p^2\tau\rho}{\alpha(z')^2} - \frac{\beta}{\alpha(z')^2}\right]V = \frac{f}{\alpha(z')^2 A}.$$

Now let $A = \rho^{-1/4}(x)$ and $z(x) = (1/L_1)\int_0^x \rho^{1/2}(x_2)\,dx_2$. Then $(A^2 z')' \equiv 0$, $A''/A(z')^2 = (\rho^{-1/4})_{xx}L_1^2/\rho^{3/4}$, $p^2\tau\rho/(\alpha(z')^2) = \tau L_1^2 p^2/\alpha$, and $\beta/(\alpha(z')^2) = \beta L_1^2/\alpha\rho$, so Eq. (9) becomes Eq. (10). ∎

In the next theorem, new differential equations are determined that are related to the Timoshenko equations by the transformation indicated in the previous lemma. Let L_1 and $z(x)$ be defined as they are in Lemma 2.1, and let $\mu_E^2 = p^2L_1^2/E$, $\mu_{kG}^2 = p^2L_1^2/kG$, and $\gamma = kAG/EI$. Since $\rho(x)$ is positive for all $x \in [0, L]$, $z(x)$ is an invertible function. Let $x(z)$ denote the inverse of $z(x)$, and let $\rho_3 = L_1^2[\rho^{-1/4}(x(z))]_{xx}/[\rho^{3/4}(x(z))]$ and $\rho_4 = \rho_x(x(z))L_1^2/[4\rho^2(x(z))]$.

THEOREM 2.1. *Let E, kG, I, A, and p^2 all be constants, let $\rho(x)$ be positive for all $x \in [0, L]$, and let $\rho_{xx}(x) \in L_2(0, L)$. Then*

$$y(x) = \rho^{-1/4}Y(z(x)) \qquad \textit{and} \qquad \Psi(x) = \rho^{-1/4}\Phi(z(x))$$

satisfy the equations

$$EI\Psi_{xx} + \left(p^2I\rho - kAG\right)\Psi = -kAGy_x + F_1, \tag{11}$$

and

$$kAGy_{xx} + p^2A\rho y = kAG\Psi_x + F_2 \tag{12}$$

if and only if Φ and Y satisfy

$$\Phi_{zz} + \frac{p^2L_1^2}{E}\Phi - \frac{L_1^2\gamma}{\rho}\Phi + \rho_3\Phi - \gamma\rho_4Y + \frac{\gamma L_1}{\rho^{1/2}}Y_z = \frac{F_1L_1^2}{EI\rho^{3/4}}, \tag{13}$$

and

$$Y_{zz} + \frac{p^2L_1^2}{kG}Y + \rho_3Y + \rho_4\Phi - \frac{L_1}{\rho^{1/2}}\Phi_z = \frac{F_2L_1^2}{kAG\rho^{3/4}}. \tag{14}$$

Proof. This theorem is an immediate consequence of Lemma 2.1. ▮

Theorem 2.1 shows how to transform the original Timoshenko differential equations into new equations so that in the new equations, coefficients involving p^2 are constant with respect to the new independent variable z. Equations (11)–(14) include generic "right hand side" terms so that Theorem 2.1 is general enough that it applies to differential equations that arise in the next section. In the case where $F_1 = F_2 \equiv 0$, Eqs. (11) and (12) are the homogeneous Timoshenko differential equations (2) and (3).

THEOREM 2.2. *Suppose $\rho(x)$ is a positive function of x such that $\rho_{xx} \in L_2(0, L)$ and $\rho_x(0) = \rho_x(L) = 0$. Let E, kG, A, I, and p^2 be positive constants. Then nontrivial Y and Φ are solutions to the boundary value* *problem*

$$\Phi_{zz} + \frac{p^2 L_1^2}{E}\Phi - \frac{L_1^2 \gamma}{\rho}\Phi + \rho_3 \Phi - \gamma \rho_4 Y + \frac{\gamma L_1}{\rho^{1/2}} Y_z = 0, \tag{15}$$

$$Y_{zz} + \frac{p^2 L_1^2}{kG} Y + \rho_3 Y + \rho_4 \Phi - \frac{L_1}{\rho^{1/2}}\Phi_z = 0, \tag{16}$$

$$\left\{ Y_z - \frac{L_1}{\rho^{1/2}}\Phi \right\}\Bigg|_{z=0,1} = 0, \qquad \textit{and} \qquad \Phi_z\big|_{z=0,1} = 0, \tag{17}$$

if and only if $y(x) = \rho^{-1/4} Y(z(x))$ *and* $\Psi(x) = \rho^{-1/4}\Phi(z(x))$ *are nontrivial solutions to the free-free Timoshenko boundary value problem* (2)–(4). *The value* p^2 *is an eigenvalue for the free-free Timoshenko boundary value problem if and only if there exist nontrivial functions* Y *and* Φ *which satisfy the differential equations* (15) *and* (16) *and the boundary conditions given in* (17).

Proof. This theorem is an easy consequence of Theorem 2.1 and the fact that when $\rho_x|_{x=0,L} = 0$, $(y_x - \psi)|_{x=0,L} = 0$ if and only if $(Y_z - L_1\Phi/\rho^{1/2})|_{z=0,1} = 0$ and $\psi_x|_{x=0,L} = 0$ if and only if $\Psi_z|_{z=0,1} = 0$. ∎

3. A FREQUENCY EQUATION

Consider the following initial value problem. Let $\alpha = L_1/\rho^{1/2}(0)$. Suppose that Y and Φ satisfy differential equations (15) and (16) and that for real c and d, they also satisfy $Y(0) = d$, $\Phi(0) = c/\alpha$, $Y_z(0) = c$, and $\Phi_z(0) = 0$. These initial conditions ensure that the boundary conditions on Y and Φ at $z = 0$ given in (17) are satisfied no matter how c and d are chosen. They are the least restrictive initial conditions on $Y(z)$ and $\Phi(z)$ which enforce the transformed boundary conditions at the left boundary point $z = 0$. Furthermore, if nontrivial c and d can be chosen so that the boundary conditions at $z = 1$ are satisfied, then by definition $L_1^2 p^2$ is an eigenvalue for the transformed boundary value problem, and by Theorem 2.2, p^2 is an eigenvalue for the free-free Timoshenko boundary value problem. In the next two lemmas, integral equations for Y and Φ are derived which are equivalent to the initial value problem discussed above. These integral equations are used to determine approximate solutions to the above initial value problem. The approximate solutions to the initial value problem make possible estimates of the values of p for which nontrivial solutions exist to the transformed boundary value problem.

LEMMA 3.1. *Let μ, c, and d be fixed constants, and let $q(z)$ and $f(z)$ be integrable. Then $w(z)$ is the solution to the initial value problem*

$$\begin{gathered} w'' + \mu^2 w = q(z)w + f(z), \\ w'(0) = c, \qquad \text{and} \qquad w(0) = d, \end{gathered} \tag{18}$$

if and only if w satisfies the integral equation

$$w(z) = c\frac{\sin(\mu z)}{\mu} + d\cos\mu z + \int_0^z \frac{\sin[\mu(z-t)]}{\mu}[q(t)w(t) + f(t)]\,dt. \tag{19}$$

Proof. The elementary proof is based on the well known technique of variation of parameters and is omitted. ∎

THEOREM 3.1. *Suppose $\rho(x)$ is a positive function of x such that $\rho_{xx} \in L_2(0, L)$ and $\rho_x(0) = \rho_x(L) = 0$. Let E, kG, A, I, and p^2 be positive constants, let $\alpha = L_1/\rho^{1/2}(0)$, and let c and d be arbitrary but fixed constants. Then $\Phi(z)$ and $Y(z)$ satisfy the differential equations* (15) *and* (16) *and the initial conditions*

$$Y(0) = d, \qquad \Phi(0) = c/\alpha, \qquad Y_z(0) = c, \qquad \Phi_z(0) = 0 \tag{20}$$

if and only if Φ and Y also satisfy the integral equations

$$\begin{aligned} \Phi = {} & \frac{c}{\alpha}\cos\mu_E z \\ & + \int_0^z \frac{\sin[\mu_E(z-t)]}{\mu_E}\left\{\left(\frac{L_1^2\gamma}{\rho} - \rho_3\right)\Phi + \gamma\rho_4 Y - \frac{\gamma L_1}{\rho^{1/2}}Y_t\right\}dt \end{aligned} \tag{21}$$

and

$$\begin{aligned} Y = {} & \frac{c\sin(\mu_{kG}z)}{\mu_{kG}} \\ & + d\cos(\mu_{kG}z) + \int_0^z \frac{\sin[\mu_{kG}(z-t)]}{\mu_{kG}}\left\{-\rho_3 Y - \rho_4\Phi + \frac{L_1}{\rho^{1/2}}\Phi_t\right\}dt. \end{aligned} \tag{22}$$

Proof. First observe that since $\rho_{xx} \in L_2(0, L)$, Theorem 2.1 shows that Y and Φ satisfy (15) and (16) if and only if $y(x) = \rho^{-1/4} Y(z(x))$ and $\Psi(x) = \rho^{-1/4}\Phi(z(x))$ satisfy the homogeneous Timoshenko differential equations. All coefficients in the Timoshenko equations (2) and (3) are continuously differentiable. Since y_x and Ψ_x must be continuous, it follows that Y_z and Ψ_z are also continuous.

Next, apply Lemma 3.1 to the transformed differential equation (15). Observe that $(L_1^2\gamma/\rho - \rho_3) \in L_2(0,1)$ and that (from the discussion of the previous paragraph) $(\gamma\rho_4 Y - [\gamma L_1/\rho^{1/2}]Y_z) \in L_2(0,1)$. Lemma 3.1 shows that the differential equation (15) and initial conditions (20) are satisfied if and only if integral equation (20) holds. Similarly, since $-\rho_3$ and $-\rho_4\Phi + [L_1/\rho^{1/2}]\Phi_z$ are in $L_2(0,1)$, Lemma 3.1 shows that (16) and (20) hold if and only if (21) holds. ∎

We will use the integral equations of Theorem 3.1 to determine approximate solutions to the initial value problem given in (15), (16), and (20). If c and d are allowed to vary over $\Re$, the solution to this initial value problem generates every solution to differential equations (15) and (16) that satisfies the left boundary conditions at $z = 0$. Consider the boundary terms

$$\frac{Y_z(1) - \left(L_1/\rho^{1/2}(1)\right)\Phi(1)}{\mu_{kG}} \quad \text{and} \quad \frac{\alpha\Phi_z(1)}{\mu_E}, \tag{23}$$

where Y and Φ are solutions to the initial value problem (15), (16), and (20). Setting the above expressions equal to zero leads to a homogeneous linear system for the arbitrary constants c and d. Values of the eigenvalue parameter p which make possible nontrivial choices for c and d correspond to the square roots of eigenvalues for the free-free Timoshenko beam. The homogeneous linear system in c and d has nontrivial solutions if and only if the determinant of the corresponding coefficient matrix is zero. This determinant will define a frequency function. The objective in this section is to determine this frequency function from estimates of the coefficients of c and d in the expressions given in (23).

The following technical fact is used many times in the estimates that follow.

LEMMA 3.2. *Suppose that $f_z \in L_\infty(0,1)$, and that δ is a real constant. Then for $z \in [0,1]$,*

$$\left|\int_0^z \sin[\mu(z-t) + \delta] f(t)\, dt\right| < \frac{\|f\|_\infty + \|f_z\|_\infty}{\mu}.$$

Proof.

$$\int_0^z \sin[\mu(z-t)+\delta] f(t)\,dt$$

$$= \frac{\cos[\mu(z-t)+\delta]}{\mu} f(t)\Big|_0^z - \int_0^z \frac{\cos[\mu(z-t)+\delta]}{\mu} f_t(t)\,dt.$$

This implies the result. ∎

LEMMA 3.3. *Suppose h and δ are real constants and that h is not equal to 0 or $L_1/\sqrt{kG}$. Let dg/dz and $\rho_{xx}(x(z)) \in L_\infty(0,1)$, and let $\rho(x) > 0$ when $x \in [0, L]$. Then*

$$\left|\int_0^z \frac{\sin[h\mu(z-t)+\delta]}{\mu} g(t) Y_t\,dt\right| < O(1/\mu)\|Y\|_\infty + O(1/\mu)\|\Phi\|_\infty$$
$$+ O(1/\mu)|c| + O(1/\mu)|d|.$$

Proof.

$$Y_z = c\cos(\mu_{kG} z) - d\mu_{kG}\sin(\mu_{kG} z)$$
$$+ \int_0^z \cos[\mu_{kG}(z-t)]$$
$$\times\left\{-\rho_3(t)Y(t) - \rho_4(t)\Phi(t) + \frac{L_1}{\rho^{1/2}(t)}\Phi_t(t)\right\} dt$$
$$\Rightarrow \int_0^z \frac{\sin[h\mu(z-t)+\delta]}{\mu} g(t) Y_t\,dt$$
$$= \int_0^z \left[\frac{\sin[h\mu(z-t)+\delta]}{\mu} g(t)\left\{c\cos(\mu_{kG} t) - d\mu_{kG}\sin(\mu_{kG} t)\right.\right.$$
$$\left.\left. + \int_0^t \cos[\mu_{kG}(t-s)][-\rho_3(s)Y(s) - \rho_4(s)\Phi(s)]\,ds\right\}\right] dt$$
$$+ \frac{\sin[h\mu(z-t)+\delta]}{\mu}$$
$$\times g(t)\left[\int_0^t \cos[\mu_{kG}(t-s)]\frac{L_1}{\rho^{1/2}(s)}\Phi_s(s)\,ds\right] dt.$$

Since by hypothesis $h\mu \neq \mu_{kG}$, a double angle formula, the assumption that $dg/dz \in L_\infty(0,1)$ and Lemma 3.2 can be used to show that the absolute value of the first integral on the right of the above equation is bounded above by

$$O(1/\mu)\|Y\|_\infty + O(1/\mu)\|\Phi\|_\infty + O(1/\mu)|c| + O(1/\mu)|d|.$$

To demonstrate a similar result for the second integral on the right of the above equation, first note that

$$\begin{aligned}\int_0^t &\cos[\mu_{kG}(t-s)]\frac{L_1}{\rho^{1/2}(s)}\Phi_s(s)\,ds \\ &= \frac{L_1\Phi(t)}{\rho^{1/2}(t)} - \frac{L_1\Phi(0)}{\rho^{1/2}(0)}\cos(\mu_{kG}t) \\ &\quad - \int_0^t\Bigg[\mu_{kG}\sin[\mu_{kG}(t-s)]\frac{L_1}{\rho^{1/2}(s)} \\ &\qquad + \cos[\mu_{kG}(t-s)]\left(\frac{L_1}{\rho^{1/2}(s)}\right)_s\Bigg]\Phi\,ds.\end{aligned}$$

Therefore, if we show that

$$\begin{aligned}\Bigg|\int_0^z &\frac{\sin[h\mu(z-t)+\delta]}{\mu}g(t) \\ &\times\left[\int_0^t \mu_{kG}\sin[\mu_{kG}(t-s)]\frac{L_1}{\rho^{1/2}(s)}\Phi(s)\,ds\right]dt\Bigg| < O\left(\frac{1}{\mu}\right),\end{aligned}$$

then the lemma follows. After changing the order of integration, the integral above may be rewritten as

$$\int_{s=0}^z\int_{t=s}^z \frac{\sin[h\mu(z-t)+\delta]}{\mu}g(t)\,\mu_{kG}\sin[\mu_{kG}(t-s)]\frac{L_1}{\rho^{1/2}(s)}\Phi(s)\,dt\,ds.$$

Again, using a double angle formula and the fact that $g'(z) \in L_\infty(0,1)$, we apply Lemma 3.2 to find that the above integral is bounded in absolute value by a function of the form $O(1/\mu)\|\Phi\|_\infty$. ∎

LEMMA 3.4. *Suppose h and δ are real constants and that h is not equal to 0 or $L_1/\sqrt{E}$. Let dg/dz and $\rho_{xx}(x(z)) \in L_\infty(0,1)$, and let $\rho(x) > 0$ when $x \in [0, L]$. Then*

$$\begin{aligned}\left|\int_0^z \frac{\sin[h\mu(z-t)+\delta]}{\mu}g(t)\Phi_t\,dt\right| &< O(1/\mu)\|Y\|_\infty + O(1/\mu)\|\Phi\|_\infty \\ &\quad + O(1/\mu)|c| + O(1/\mu)|d|.\end{aligned}$$

Proof. The proof of Lemma 3.4 is similar to the proof of Lemma 3.3 and is therefore omitted. ∎

In the next theorem, we show that the infinity norms of Y and Φ remain finite as p approaches infinity.

LEMMA 3.5. *Suppose $E \neq kG$, that $\rho_{xx} \in L_\infty(0, L)$, and that $\rho(x) > 0$ for all $x \in [0, 1]$. Then*

$$\|\Phi\|_\infty \leq O(1)|c| + O(1/p)|d|$$

and

$$\|Y\|_\infty \leq O(1/p)|c| + O(1)|d|.$$

Proof. The integral equations for Φ and Y together with Lemma 3.3 and Lemma 3.4 imply that

$$\|\Phi\|_\infty \leq O(1)|c| + O(1/p)|d| + O(1/p)\|\Phi\|_\infty + O(1/p)\|Y\|_\infty \tag{24}$$

and

$$\|Y\|_\infty \leq O(1/p)|c| + O(1)|d| + O(1/p)\|\Phi\|_\infty + O(1/p)\|Y\|_\infty. \tag{25}$$

Inequality (24) implies that

$$(1 - O(1/p))\|\Phi\|_\infty \leq O(1)|c| + O(1/p)|d| + O(1/p)\|Y\|_\infty.$$

For p large enough, this inequality implies that

$$\|\Phi\|_\infty \leq O(1)|c| + O(1/p)|d| + O(1/p)\|Y\|_\infty. \tag{26}$$

Similarly, from inequality (25), we find that for p large enough

$$\|Y\|_\infty \leq O(1/p)|c| + O(1)|d| + O(1/p)\|\Phi\|_\infty. \tag{27}$$

The theorem follows from inequalities (26) and (27). ∎

In the next theorem, we calculate estimates of the coefficients of c and d in the functions and $(Y_z - \Phi L_1/\rho^{1/2}(z))/\mu_{kG}$ and $\alpha\Phi_z/\mu_E$.

LEMMA 3.6. *Suppose $E \neq kG$, that $\rho_{xx} \in L_\infty(0, L)$ and $\rho(x) > 0$ for all $x \in [0, L]$, and that $\alpha = L_1/\rho^{1/2}(0)$. Then*

$$\left\| \frac{Y_z(z) - \left(L_1/\rho^{1/2}(z)\right)\Phi(z)}{\mu_{kG}} - c\,\frac{\cos(\mu_{kG} z) - (\rho^{1/2}(0)/\rho^{1/2}(z))\cos(\mu_E z)}{\mu_{kG}} + d\sin(\mu_{kG} z) \right\|_\infty$$

and

$$\left\| \frac{\Phi_z(z)}{\mu_E} + \frac{c}{\alpha}\sin(\mu_E z) \right\|_\infty \le O(1/p)|c| + O(1/p)|d|.$$

Proof. Integral equations for Y_z/μ_{kG} and Φ_z/μ_E and can be determined from the integral equations for Y and Φ. The proof follows from the integral equations for Y and Φ and Lemmas 3.3, 3.4, and 3.5. ∎

Lemma 3.6 facilitates the derivation of a frequency equation for the free-free Timoshenko beam with variable density $\rho(x)$.

THEOREM 3.2. *Let E, kG, A, and I be positive constants such that $E \neq kG$. Let $\rho(x) > 0$ for all $x \in [0, L]$, let $\rho_x(0) = \rho_x(L) = 0$, and suppose $\rho_{xx}(x) \in L_\infty(0, L)$. Then p^2 is an eigenvalue for the free-free Timoshenko beam if and only if*

$$F(p) \equiv \sin \mu_{kG} \sin \mu_E + \sin(\mu_E)\,\delta_{1,2} + \sin(\mu_{kG})\,\delta_{2,1} + \delta = 0, \quad (28)$$

where the functions $\delta_{1,2}(p)$, and $\delta_{2,1}(p)$ are $O(1/p)$ and $\delta(p)$ is $O(1/p^2)$.

Proof. We seek to determine the values of p^2 for which there exist nontrivial functions Φ and Y that solve the transformed differential equations and all transformed boundary conditions, including those at $z = 1$. To determine all such solutions, we seek nontrivial solutions to the initial value problem in (15), (16), and (20) which also solve the transformed boundary conditions at $z = 1$. Theorem 2.2 shows that the values of p^2 which admit nontrivial solutions when boundary conditions at $z = 1$ are imposed are the eigenvalues of the free-free Timoshenko beam.

Lemma 3.6 implies that for any choice of c and d, solutions Y and Φ to the initial value problem (15), (16), and (20) satisfy

$$\frac{Y_z(1) - \left[L_1/\rho^{1/2}(1)\right]\Phi(1)}{\mu_{kG}} = \delta_{1,1}c + \left[\sin(\mu_{kG}) - \delta_{1,2}\right]d \quad (29)$$

and

$$\frac{\alpha\Phi_z}{\mu_E} = [-\sin \mu_E - \delta_{2,1}]c - \delta_{2,2}d, \quad (30)$$

where the $\delta_{i,j}$ are all $O(1/p)$ functions. Equations (29) and (30) imply that the right boundary conditions, i.e., the conditions in (20) when $z = 1$, may be satisfied if and only if

$$\begin{bmatrix} \delta_{1,1} & -\sin \mu_{kG} - \delta_{1,2} \\ -\sin \mu_E - \delta_{2,1} & -\delta_{2,2} \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This linear system can be nontrivially solved if and only if

$$\sin \mu_{kG} \sin \mu_E + \sin(\mu_E)\,\delta_{1,2} + \sin(\mu_{kG})\,\delta_{2,1} + O(1/p^2) = 0.$$

∎

4. ROOTS OF THE FREQUENCY FUNCTION

In this section, four results are presented. Theorem 4.1 shows that all roots of the frequency equation F must occur near roots of $\sin(\mu_E) \cdot \sin(\mu_{kG})$. Theorem 4.2 shows that near each root of $\sin(\mu_E) \cdot \sin(\mu_{kG})$ which is isolated from neighboring roots, there must exist at least one root of F. In Lemma 4.1, an approximation for $\partial F / \partial p$ is calculated. Theorems 4.1, 4.2, and Lemma 4.1 facilitate the proof of Theorem 4.3, in which it is shown that *exactly one* root of F occurs near isolated roots of $\sin(\mu_E) \cdot \sin(\mu_{kG})$.

THEOREM 4.1. *Let*

$$F(p) \equiv \sin \mu_{kG} \sin \mu_E + \sin(\mu_E)\,\delta_{1,2} + \sin(\mu_{kG})\,\delta_{2,1} + \delta,$$

where $\delta_{1,2}$ and $\delta_{2,1}$ are $O(1/p)$ and δ is $O(1/p^2)$. Let $\bar{c} = \max\{\sqrt{E}, \sqrt{kg}\}$, and suppose p is a root of the frequency function $F(p)$. Then there exists a root of the function $\sin(\mu_{kG})\sin(\mu_E)$, say $\hat{p}$, such that $p - \hat{p} = \epsilon_p$, where

$$|\epsilon_p| < \frac{\bar{c}}{L_1} \sin^{-1}\left\{ \frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{\left(|\delta_{1,2}| + |\delta_{2,1}|\right)^2}{4} + |\delta|} \right\}.$$

(*Thus*, $|\epsilon_p|$ *is* $O(1/p)$.)

Proof. If p is a root of the frequency equation F, then

$$\sin\left(\frac{pL_1}{\sqrt{kG}}\right)\sin\left(\frac{pL_1}{\sqrt{E}}\right) = -\sin\left(\frac{pL_1}{\sqrt{E}}\right)\delta_{1,2} - \sin\left(\frac{pL_1}{\sqrt{kG}}\right)\delta_{2,1} - \delta. \quad (31)$$

If either $\sin(pL_1/\sqrt{kG})$ or $\sin(pL_1/\sqrt{E})$ are zero, then the result follows. Suppose neither $\sin(pL_1/\sqrt{kG})$ nor $\sin(pL_1/\sqrt{E})$ is zero. Assume for now that

$$\left| \frac{\sin\left(pL_1/\sqrt{E} \right)}{\sin\left(pL_1/\sqrt{kG} \right)} \right| \leq 1 \tag{32}$$

and divide both sides of Eq. (31) above by $\sin(pL_1/\sqrt{kG})$. It follows that

$$\begin{aligned}
\left|\sin\left(pL_1/\sqrt{E} \right)\right| &= \left| \delta_{1,2} \frac{\sin\left(pL_1/\sqrt{E} \right)}{\sin\left(pL_1/\sqrt{kG} \right)} + \delta_{2,1} + \frac{\delta}{\sin\left(pL_1/\sqrt{kG} \right)} \right| \\
&\leq |\delta_{1,2}| + |\delta_{2,1}| + \frac{|\delta|}{\left|\sin\left(pL_1/\sqrt{E} \right)\right|} \\
&\Rightarrow \sin^2\left(pL_1/\sqrt{E} \right) < (|\delta_{1,2}| + |\delta_{2,1}|) \left|\sin\left(pL_1/\sqrt{E} \right)\right| + |\delta|.
\end{aligned}$$

Let $s = |\sin(pL_1/\sqrt{E})|$, $b = |\delta_{1,2}| + |\delta_{2,1}|$, and $c = |\delta|$. Then $s > 0$ and $s^2 - bs - c < 0$. This implies that

$$0 < s < \frac{b + \sqrt{b^2 + 4c}}{2},$$

and hence that,

$$\left|\sin\left(pL_1/\sqrt{E} \right)\right| < \frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{\left(|\delta_{1,2}| + |\delta_{2,1}|\right)^2}{4} + |\delta|}. \tag{33}$$

Let M be an integer chosen such that $|[L_1/\sqrt{E}]p - M\pi| \leq \pi/2$, and let $\epsilon_p = p - [\sqrt{E}/L_1]M\pi$. Then from inequality (33) it follows that

$$\begin{aligned}
\left| \sin\left(\frac{pL_1}{\sqrt{E}} \right) \right| &= \left| \sin\left(\frac{L_1 \epsilon_p}{\sqrt{E}} \right) \right| = \sin\left(\frac{L_1 |\epsilon_p|}{\sqrt{E}} \right) \\
&< \frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{\left(|\delta_{1,2}| + |\delta_{2,1}|\right)^2}{4} + |\delta|} \\
&\Rightarrow |\epsilon_p| < \frac{\sqrt{E}}{L_1} \sin^{-1} \left\{ \frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{\left(|\delta_{1,2}| + |\delta_{2,1}|\right)^2}{4} + |\delta|} \right\}.
\end{aligned}$$

Thus, there is some integer M such that $p = \epsilon_p + [\sqrt{E}/L_1]M\pi$ where

$$|\epsilon_p| < \frac{\sqrt{E}}{L_1}\sin^{-1}\left\{\frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{(|\delta_{1,2}| + |\delta_{2,1}|)^2}{4} + |\delta|}\right\}.$$

The argument above was carried out under the assumption that (32) holds. If instead $|\sin(pL_1/\sqrt{E})/\sin(pL_1/\sqrt{kG})| > 1$, the same argument given above holds in this case provided E and kG are interchanged. In either case, it follows that there exists a root of $\sin(\mu_{kG})\sin(\mu_E)$, say $\hat{p} = \sqrt{E}M\pi/L_1$ or $\sqrt{kG}M\pi/L_1$, such that $p - \hat{p} = \epsilon_p$ and

$$|\epsilon_p| < \frac{\bar{c}}{L_1}\sin^{-1}\left\{\frac{|\delta_{1,2}| + |\delta_{2,1}|}{2} + \sqrt{\frac{(|\delta_{1,2}| + |\delta_{2,1}|)^2}{4} + |\delta|}\right\}.$$

∎

THEOREM 4.2. *Suppose $\hat{p}$ and $\tilde{p}$ are two adjacent roots of the function $\sin(\mu_{kG})\cdot\sin(\mu_E)$ and that $\hat{p}$ is a simple root not closer to any other root of this function than it is to $\tilde{p}$. Let $\bar{c} = \max\{\sqrt{E}, \sqrt{kG}\}$, $\underline{c} = \min\{\sqrt{E}, \sqrt{kG}\}$ and let $\bar{p}$ be the largest root of $\sin(\mu_{kG})\sin(\mu_E)$ which satisfies $\bar{p} < \min\{\hat{p}, \tilde{p}\}$. Let*

$$\Delta p = \sin^{-1}\left\{\frac{\bar{c}}{\underline{c}}\frac{\pi}{2}\sup_{p>\bar{p}}|\delta_{1,2}(p)| + \sup_{p>\bar{p}}|\delta_{2,1}(p)| + \sup_{p>\bar{p}}\frac{|\delta|}{\sin(\bar{p}^{-1})}\right\}\frac{\bar{c}}{L_1},$$

and suppose that $|\hat{p} - \tilde{p}| > 2\Delta p + (\bar{c}/L_1)(1/\bar{p})$. Then there is at least one root of the characteristic equation (28) *in the interval $[\hat{p} - \Delta p, \hat{p} + \Delta p]$. Furthermore, this interval must contain an odd number of roots of the characteristic equation.*

Remark. The quantity Δp defined above is $O(1/\bar{p})$, since all of the δ's appearing in its definition are $O(1/\bar{p})$. Thus, the hypothesis above requires that $\hat{p}$ and $\tilde{p}$ be separated by a distance which is $O(1/\bar{p})$.

Proof Sketch. The proof follows from the basic observation that $\sin(\mu_{kG})\cdot\sin(\mu_E)$ is strictly monotonic near its isolated roots, and that $F = \sin(\mu_{kG})\cdot\sin(\mu_E) + O(1/p)$. For $\hat{p}$ large enough, $F(p)$ must change sign near where $\sin(\mu_{kG})\cdot\sin(\mu_E)$ changes sign, i.e., in the interval $[\hat{p} - \Delta p, \hat{p} + \Delta p]$. See Geist [8, pp. 164–168] for a detailed presentation of this proof. ∎

An estimate of $\partial F/\partial p$ derived in the next lemma is an important step toward proving that for each isolated root of $\sin(\mu_E)\cdot\sin(\mu_{kG})$, there is *exactly* one zero of F nearby.

LEMMA 4.1. *Let F be the frequency function defined in* (28). *Then*

$$\frac{\partial F}{\partial p} = \frac{L_1}{\sqrt{E}}\cos(\mu_E)\cdot\sin(\mu_{kG}) + \frac{L_1}{\sqrt{kG}}\cos(\mu_{kG})\cdot\sin(\mu_E) + r(p), \tag{34}$$

where $r(p) = O(1/p)$.

Proof Sketch. If δ, $\delta_{1,2}$, and $\delta_{2,1}$ were all known exactly, it might be possible to calculate $\partial F/\partial p$ by direct differentiation. Unfortunately, only order estimates for the δ's are known; δ, $\delta_{1,2}$, and $\delta_{2,1}$ are not known explicitly. However, $\partial\mathscr{F}/\partial p$ can be estimated using the formula

$$-\frac{\partial F}{\partial p} = \det\begin{bmatrix} \begin{matrix}\text{coef.}\\ \text{of } c\end{matrix}\left[\dfrac{Y_z(1) - \left(L_1/\rho^{1/2}(1)\right)\Phi(1)}{\mu_{kG}}\right], & \begin{matrix}\text{coef.}\\ \text{of } d\end{matrix}\left[\dfrac{Y_z(1) - \left(L_1/\rho^{1/2}(1)\right)\Phi(1)}{\mu_{kG}}\right] \\ \begin{matrix}\text{coef.}\\ \text{of } c\end{matrix}\dfrac{\partial}{\partial p}\left[\dfrac{\alpha\Phi_z(1)}{\mu_E}\right], & \begin{matrix}\text{coef.}\\ \text{of } d\end{matrix}\dfrac{\partial}{\partial p}\left[\dfrac{\alpha\Phi_z(1)}{\mu_E}\right] \end{bmatrix}$$
$$+\det\begin{bmatrix} \begin{matrix}\text{coef.}\\ \text{of } c\end{matrix}\dfrac{\partial}{\partial p}\left[\dfrac{Y_z(1) - \left(L_1/\rho^{1/2}(1)\right)\Phi(1)}{\mu_{kG}}\right], & \begin{matrix}\text{coef.}\\ \text{of } d\end{matrix}\dfrac{\partial}{\partial p}\left[\dfrac{Y_z(1) - \left(L_1/\rho^{1/2}(1)\right)\Phi(1)}{\mu_{kG}}\right] \\ \begin{matrix}\text{coef.}\\ \text{of } c\end{matrix}\left[\dfrac{\alpha\Phi_z(1)}{\mu_E}\right], & \begin{matrix}\text{coef.}\\ \text{of } d\end{matrix}\left[\dfrac{\alpha\Phi_z(1)}{\mu_E}\right] \end{bmatrix}, \tag{35}$$

where in the expression above each entry in each matrix is a coefficient of either c or d. Let $\cdot$ denote differentiation with respect to p. Integral equations for $\dot{Y}_z$, $\dot{\Phi}$, and $\dot{\Phi}_z$ can be determined from the integral equations for Y and Φ. (It follows from Theorem 2.2 that Y and Φ, which satisfy the initial value problem (15), (16), and (20) may be written as $Y(z) = [\rho(x(z))]^{1/4}y(x(z))$ and $\Phi(z) = [\rho(x(z))]^{1/4}\Psi(x(z))$, where $y(x)$ and $\Psi(x)$ satisfy the original Timoshenko differential equations (2) and (3) and the initial conditions $y(0) = \rho^{-1/4}(0)d$, $y_x(0) = [\rho^{1/4}(0)/L_1]c$, $\Psi(0) = [\rho^{1/4}(0)/L_1]c$, and $\Psi_x(0) = 0$. For each $x \in [0, L]$, y, y_x, Ψ, and Ψ_x are analytic functions of p. See [4, p. 37]. This implies that for each $z \in [0, 1]$, Y_z, Φ_z, and Φ are analytic functions of p, and so differentiation with respect to the eigenvalue parameter p is allowed.) Estimates for $\dot{Y}_z$, $\dot{\Phi}$, and $\dot{\Phi}_z$ can be obtained by first showing that the infinity norms of these functions are bounded. Once this is demonstrated, one can generate estimates for $\dot{Y}_z$, $\dot{\Phi}$, and $\dot{\Phi}_z$ accurate to within an error that is $O(1/p)$ in the same way that estimates for Y, Y_z, and Φ_z were produced in Section 3. Then, estimates for $\partial F/\partial p$ can be calculated using Eq. (35). For the details of this calculation, see Geist [8, pp. 173–189]. ∎

We focus now on proving that for each isolated root of $\sin(\mu_E)\cdot \sin(\mu_{kG})$, there is exactly one zero of F nearby. In particular, we demonstrate that $|\partial F/\partial p| > 0$ near roots of $\sin(\mu_E)\cdot\sin(\mu_{kG})$ that are not too close to one another.

THEOREM 4.3. *Suppose $\hat{p}$ and $\tilde{p}$ are two adjacent roots of the function $\sin(\mu_{kG})\cdot\sin(\mu_E)$. Suppose $\hat{p}$ is a simple root of $\sin(\mu_{kG})\cdot\sin(\mu_E)$ not closer to any other root of this function than it is to $\tilde{p}$. Let $\bar{c} = \max\{\sqrt{E}, \sqrt{kG}\}$, $\underline{c} = \min\{\sqrt{E}, \sqrt{kG}\}$ and let $\bar{p}$ be the largest root of $\sin(\mu_{kG})\cdot\sin(\mu_E)$ which satisfies $\bar{p} < \min\{\hat{p}, \tilde{p}\}$. Let*

$$\Delta p = \sin^{-1}\left\{\frac{\bar{c}}{\underline{c}}\frac{\pi}{2}\sup_{p>\bar{p}}|\delta_{1,2}(p)| + \sup_{p>\bar{p}}|\delta_{2,1}(p)| + \sup_{p>\bar{p}}\frac{|\delta|}{\sin(\bar{p}^{-1})}\right\}\frac{\bar{c}}{L_1},$$

$$\Gamma = \sup_{p>\bar{p}-\Delta p}\left\{\frac{\bar{c}}{L_1}\sin^{-1}\left\{\frac{|\delta_{1,2}|+|\delta_{2,1}|}{2} + \sqrt{\frac{(|\delta_{1,2}|+|\delta_{2,1}|)^2}{4} + |\delta|}\right\}\right\}, \tag{36}$$

and

$$r(p) = \frac{\partial F}{\partial p} - \frac{L_1}{\sqrt{E}}\cos(\mu_E)\cdot\sin(\mu_{kG}) - \frac{L_1}{\sqrt{kG}}\cos(\mu_{kG})\cdot\sin(\mu_E).$$

Let $\hat{p}$ be large enough so that $\Gamma < \underline{c}\pi/[4L_1]$. If for $p \in [\hat{p}-\Gamma, \hat{p}+\Gamma]$

$$|\hat{p}-\tilde{p}| > \max\left\{2\Delta p + \frac{\bar{c}}{L_1}\frac{1}{\bar{p}}, \frac{\bar{c}}{L_1}\frac{\pi}{2}\left|\tan\left(\frac{L_1\Gamma}{\underline{c}}\right)\right| + \frac{\sqrt{EkG}}{L_1^2}\frac{\pi}{2}\frac{|r(p)|}{\cos(\Gamma L_1/\underline{c})} + \Gamma\right\}, \tag{37}$$

then there is exactly one root of the characteristic function F in the $[\hat{p} - \Gamma, \hat{p} + \Gamma]$.

Remark. r is $O(1/p)$ and Γ and Δp are $O(1/\bar{p})$.

Proof. Theorem 4.1 demonstrates that all roots of F occur an $O(1/p)$ distance from the zeros of $\sin(\mu_{kG})\sin(\mu_E)$. In particular, it follows from Theorem 4.1 that any root of F nearer to $\hat{p}$ than it is to any other root of $\sin(\mu_{kG}) \cdot \sin(\mu_E)$ must occur in the narrow interval $[\hat{p} - \Gamma, \hat{p} + \Gamma]$. Theorem 4.2 guarantees that in this interval, there is at least one root of F. We prove now that $|\partial F/\partial p| > 0$ throughout $[\hat{p} - \Gamma, \hat{p} + \Gamma]$ when the hypotheses of this theorem hold.

Lemma 4.1 shows that

$$\frac{\partial F}{\partial p} = \frac{L_1}{\sqrt{E}} \cos(\mu_E) \cdot \sin(\mu_{kG}) + \frac{L_1}{\sqrt{kG}} \cos(\mu_{kG}) \cdot \sin(\mu_E) + r(p),$$

where $r(p)$ is an $O(1/p)$ function. Let $p \in [\hat{p} - \Gamma, \hat{p} + \Gamma]$. Without loss of generality, suppose $\hat{p}$ is a zero of $\sin(\mu_E)$. Hypothesis (37) implies that

$$\frac{2L_1^2}{\pi\sqrt{EkG}} \left[|\hat{p} - \tilde{p}| - \Gamma \right] > \frac{L_1}{\sqrt{kG}} \left| \tan\left(\frac{L_1}{\sqrt{E}} \Gamma \right) \right| + \frac{|r(p)|}{\cos\left((L_1/\sqrt{E})\Gamma \right)}. \quad (38)$$

Let p_{kG} be a zero of $\sin(\mu_{kG})$ at least as near to p as is any other root of $\sin(\mu_E)$. This implies that $|(p - p_{kG})L_1/\sqrt{kG}| \leq \pi/2$, and hence that

$$\begin{aligned} \frac{L_1}{\sqrt{E}} |\sin(\mu_{kG})| &= \frac{L_1}{\sqrt{E}} \left| \sin\left(\frac{L_1}{\sqrt{kG}} (p - p_{kG}) \right) \right| \geq \frac{L_1}{\sqrt{E}} \frac{2}{\pi} \frac{L_1}{\sqrt{kG}} |p - p_{kG}| \\ &\geq \frac{2L_1^2}{\pi\sqrt{EkG}} |\hat{p} - p_{kG} + p - \hat{p}| \\ &\geq \frac{2L_1^2}{\pi\sqrt{EkG}} \left[|\hat{p} - p_{kG}| - |p - \hat{p}| \right] \\ &\geq \frac{2L_1^2}{\pi\sqrt{EkG}} \left[|\hat{p} - \tilde{p}| - \Gamma \right]. \end{aligned} \quad (39)$$

$p \in [\hat{p} - \Gamma, \hat{p} + \Gamma]$, $\hat{p}$ a zero of $\tan(\mu_E)$, and $\Gamma L_1 / \sqrt{E} < \pi/4$ imply that

$$\begin{aligned} &\frac{L_1}{\sqrt{kG}} \left| \tan\left(\frac{L_1}{\sqrt{E}} \Gamma \right) \right| + \frac{|r(p)|}{\cos\big((L_1/\sqrt{E}\,)\Gamma \big)} \\ &\quad = \frac{L_1}{\sqrt{kG}} \left| \tan\left(\frac{L_1}{\sqrt{E}} (\hat{p} \pm \Gamma) \right) \right| + \frac{|r(p)|}{\cos\big((L_1/\sqrt{E}\,)(\hat{p} \pm \Gamma) \big)} \\ &\quad \geq \frac{L_1}{\sqrt{kG}} |\tan \mu_E| + \frac{|r(p)|}{|\cos \mu_E|} \geq \frac{L_1}{\sqrt{kG}} |\tan \mu_E| \, |\cos \mu_{kG}| + \frac{r(p)}{|\cos \mu_E|}. \end{aligned} \tag{40}$$

Therefore, from (38), (39), and (40) it follows that

$$\begin{aligned} &\frac{L_1}{\sqrt{E}} |\sin \mu_{KG}| \geq \frac{L_1}{\sqrt{kG}} |\tan \mu_E \cos \mu_{kG}| + \frac{|r(p)|}{|\cos \mu_E|} \\ &\quad \Rightarrow \left| \frac{L_1}{\sqrt{E}} \sin \mu_{kG} \cos \mu_E \right| > \left| \frac{L_1}{\sqrt{kG}} \sin \mu_E \cos \mu_{kG} \right| + |r(p)| \\ &\quad \Rightarrow \left| \frac{\partial F}{\partial p}(p) \right| > 0, \end{aligned}$$

as desired.

Thus, $|\partial F / \partial p| > 0$ for all $p \in [\hat{p} - \Gamma, \hat{p} + \Gamma]$, which in turn implies that there is at most one root of F in this interval. Since it has been established that there is at least one root in this interval, the theorem follows when $\hat{p}$ is a zero of $\sin(\mu_E)$. The proof for the case where $\hat{p}$ is a zero of $\sin(\mu_{kG})$ is identical to the proof above, except that the roles of E and kG are interchanged. ∎

5. THE ASYMPTOTIC FORMULAS

We carry out the following steps in order to sharpen our estimates of the eigenvalues. First, we define a new density function $\hat{\rho}$ in terms of ρ. Let $\rho_0 = [\frac{1}{L} \int_0^L \rho^{1/2}(x)\, dx]^2$ and $\tilde{\rho} = \rho(x) - \rho_0$; then define $\hat{\rho} = \rho_0 + t\tilde{\rho}$, where t is an auxiliary parameter which is allowed to vary from 0 to 1.

Note that when $t = 0$, $\hat{\rho}$ is the constant ρ_0, and when $t = 1$, $\hat{\rho} \equiv \rho(x)$. Let $\hat{\Phi}$ and $\hat{Y}$ be solutions to the boundary value problem given in (15)–(17), except z, ρ, ρ_3, ρ_4, and L_1 are replaced by $\hat{z}$, $\hat{\rho}$, $\hat{\rho}_3$, $\hat{\rho}_4$, and $\hat{L}_1$, respectively, where $\hat{z}$, $\hat{\rho}_3$, $\hat{\rho}_4$, and the constant $\hat{L}_1$ are defined as are z, ρ_3, ρ_4, and L_1 except that $\hat{\rho}$ replaces ρ in their definitions. The resulting boundary value problem for $\hat{\Phi}$ and $\hat{Y}$ corresponds to the transformed boundary value problem (see Theorem 2.2) one obtains from the original Timoshenko differential equations and boundary conditions when $\rho(x)$ is taken to be $\hat{\rho}(x; t)$.

It will prove useful to view the transformed differential equations for $\hat{\Phi}$ and $\hat{Y}$ as a single vector equation. Define

$$\hat{U}(\hat{z}) = \begin{pmatrix} \hat{\Phi}(\hat{z}) \\ \hat{Y}(\hat{z}) \end{pmatrix}.$$

Let

$$B_1 = \begin{pmatrix} \dfrac{1}{E} & 0 \\ 0 & \dfrac{1}{kG} \end{pmatrix}, \qquad Q = \begin{pmatrix} \dfrac{-\hat{L}_1^2 \gamma}{\hat{\rho}} + \hat{\rho}_3 & -\gamma \hat{\rho}_4 \\ \hat{\rho}_4 & \hat{\rho}_3 \end{pmatrix} \tag{41}$$

$$S = \begin{pmatrix} 0 & \gamma \\ -1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mu^2 = \hat{L}_1^2 p^2. \tag{42}$$

Then the boundary value problem (15)–(17), where ρ is now replaced by $\hat{\rho}$, may be written in vector notation as

$$\hat{U}_{\hat{z}\hat{z}} + \mu^2 B_1 \hat{U} + Q\hat{U} + \frac{\hat{L}_1}{\hat{\rho}^{1/2}} S \hat{U}_{\hat{z}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{43}$$

$$\left\{ \hat{Y}_z - \frac{\hat{L}_1}{\hat{\rho}^{1/2}} \hat{\Phi} \right\} \bigg|_{\hat{z} = 0,1} = 0, \qquad \text{and} \qquad \hat{\Phi}_{\hat{z}} \big|_{\hat{z} = 0,1} = 0. \tag{44}$$

Suppose the transformed eigenvalue parameter, $\mu^2 = \hat{L}_1^2 p^2$, is chosen so that a nontrivial solution $\hat{U}(\hat{z})$ exists to the above boundary value problem. Then by Theorem 2.2, if $\hat{u}$ is defined as

$$\hat{u}(x) \equiv \begin{pmatrix} \hat{\Psi}(x) \\ \hat{y}(x) \end{pmatrix} = \hat{\rho}^{-1/4} \hat{U}(\hat{z}(x)) = \hat{\rho}^{-1/4}(x) \begin{pmatrix} \hat{\Phi}(\hat{z}(x)) \\ \hat{Y}(\hat{z}(x)) \end{pmatrix},$$

then $\hat{u}$ must satisfy

$$\begin{pmatrix} \dfrac{\partial}{\partial x} EI \dfrac{\partial}{\partial x} + \left(p^2 I \hat{\rho} - kAG \right) & kAG \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} kAG & \dfrac{\partial}{\partial x} kAG \dfrac{\partial}{\partial x} + p^2 A \hat{\rho} \end{pmatrix} \hat{u}(x) = \vec{0} \quad (45)$$

$$\left[\hat{y}_x - \hat{\Psi} \right] \Big|_{x=0,L} = 0, \qquad \hat{\Psi}|_{x=0,L} = 0. \quad (46)$$

Thus, if $\hat{L}_1^2 p^2 = \mu^2$ is an eigenvalue for the transformed boundary value problem (43)–(44) and $\hat{U}(\hat{z})$ is the corresponding eigenfunction, then p^2 is an eigenvalue and $\hat{u}(x) = \hat{\rho}^{-1/4} \hat{U}(\hat{z}(x))$ is the corresponding eigenfunction for the free-free Timoshenko beam with density $\hat{\rho}$.

Our method for sharpening the estimates of the eigenvalues of the Timoshenko beam will rely on determining the derivative of μ^2 with respect to the auxiliary parameter t, introduced in the definition of $\hat{\rho}$. The approach may be summarized as follows. Let μ^2 be an eigenvalue for the transformed boundary value problem (43)–(44) and let $\hat{U}(\hat{z})$ be a corresponding eigenfunction. The differential equation and boundary conditions for $\hat{U}(\hat{z})$ are differentiated with respect to $\hat{\rho}$ in the direction of $\tilde{\rho}$. This differentiation of (43)–(44) will lead to a new, non-homogeneous boundary value problem. The right hand side of this new boundary value problem will contain $d_{\hat{\rho}} \mu^2[\tilde{\rho}]$, which we will show is equal to $d\mu^2/dt$. Since the new boundary value problem comes from differentiating the differential equations and boundary conditions (43)–(44), a boundary value problem with a known solution $\hat{U}(\hat{z}) \not\equiv 0$, it follows that a nontrivial solution to the new, non-homogeneous problem exists, and must be equal to $d_{\hat{\rho}} \mu^2[\tilde{\rho}]$. Using Theorem 2.1, the new non-homogeneous boundary value problem for $d_{\hat{\rho}} \mu^2[\tilde{\rho}]$ may be written as a non-homogeneous Timoshenko boundary value problem which must also have a nontrivial solution. The fact that the non-homogeneous Timoshenko boundary value problem has a nontrivial solution implies that the right hand side of the non-homogeneous problem must satisfy a certain orthogonality requirement (see Lemma 5.1 below). This orthogonality condition allows us to determine $d\mu^2/dt = d_{\hat{\rho}} \mu^2[\tilde{\rho}]$ in terms of the transformed eigenfunction $\hat{U}(\hat{z})$; i.e., from the orthogonality condition it follows that

$$\frac{d(\mu^2)}{dt} = G\left(\hat{U}(\hat{z}(x))\right).$$

Integrating both sides of the above equation gives

$$\mu^2|_{t=1} - \mu^2|_{t=0} = \int_0^1 G\big(\hat{U}(\hat{z}(x))\big)\, dt. \tag{47}$$

Sharp estimates of $\mu^2|_{t=0}$ can be obtained by applying formulas from Geist and McLaughlin [10]. In particular, when $t = 0$, $\hat{\rho}$ is ρ_0, a constant. Therefore $\mu^2|_{t=0}$ may be determined from the formulas for the eigenvalues of the uniform Timoshenko beam. From results in the previous five sections, $G(\hat{U}(\hat{z}(x)))$ can be determined to within an error that converges to zero as μ (and p) get large. Thus, Eq. (47) provides a means for obtaining sharp estimates for μ^2 when $t = 1$. From these estimates of μ^2, we will obtain asymptotic formulas for the free-free Timoshenko eigenvalues when mass density is $\rho(x)$.

Our next goal then is to calculate an expression for $d\mu^2/dt$. Suppose $\rho_{xx} \in L_\infty[0, L]$. For these calculations, we first assume that for all such ρ,

(i) $d_{\hat{\rho}}\mu^2[\tilde{\rho}]$ exists, that

(ii) $d_{\hat{\rho}}\hat{U}[\tilde{\rho}]$, $d_{\hat{\rho}}(\hat{U}_{\hat{z}})[\tilde{\rho}]$, $d_{\hat{\rho}}(\hat{U}_{\hat{z}\hat{z}})[\tilde{\rho}]$ all exist, and that

(iii) $(d_{\hat{\rho}}\hat{U}[\tilde{\rho}])_{\hat{z}\hat{z}} = d_{\hat{\rho}}(\hat{U}_{\hat{z}\hat{z}})[\tilde{\rho}]$ and $(d_{\hat{\rho}}\hat{U}[\tilde{\rho}])_{\hat{z}} = d_{\hat{\rho}}(\hat{U}_{\hat{z}})[\tilde{\rho}]$.

Later, we will give conditions which guarantee that assumptions (i), (ii), and (iii) are valid. As indicated in the discussion above, we formally differentiate the differential equations and boundary conditions in (43)–(44) with respect to $\hat{\rho}$ in the direction of $\tilde{\rho}$. We use assumptions (i), (ii), and (iii) from above to obtain the differential equation and boundary conditions that $d_{\hat{\rho}}\hat{U}[\tilde{\rho}]$ must satisfy. We find that

$$\begin{aligned}\Big(d_{\hat{\rho}}\hat{U}[\tilde{\rho}]\Big)_{\hat{z}\hat{z}} + \mu^2 B_1\Big(d_{\hat{\rho}}\hat{U}[\tilde{\rho}]\Big) + Q\Big(d_{\hat{\rho}}\hat{U}[\tilde{\rho}]\Big) + \frac{\hat{L}_1}{\hat{\rho}^{1/2}} S\Big(d_{\hat{\rho}}\hat{U}[\tilde{\rho}]\Big)_{\hat{z}} \\ = -\Big(d_{\hat{\rho}}\mu^2[\tilde{\rho}]\Big)B_1\hat{U} - \Big(d_{\hat{\rho}}Q[\tilde{\rho}]\Big)\hat{U} - d_{\hat{\rho}}\Big(\hat{L}_1/\hat{\rho}^{1/2}\Big)[\tilde{\rho}]S\hat{U}_{\hat{z}},\end{aligned}$$

and that

$$\Big[\Big(d_{\hat{\rho}}\hat{Y}[\tilde{\rho}]\Big)_{\hat{z}} - \Big(\hat{L}_1/\hat{\rho}^{1/2}\Big)\Big(d_{\hat{\rho}}\hat{\Phi}[\tilde{\rho}]\Big) - \Big(d_{\hat{\rho}}\Big(\hat{L}_1/\hat{\rho}^{1/2}\Big)[\tilde{\rho}]\Big)\hat{\Phi}\Big]\Big|_{\hat{z}=0,1} = 0,$$

$$\Big(d_{\hat{\rho}}\hat{\Phi}[\tilde{\rho}]\Big)\Big|_{\hat{z}=0,1} = 0.$$

When $d_{\hat\rho}\mu^2[\tilde\rho]$ exists, it follows that

$$\begin{aligned} d_{\hat\rho}\mu^2[\tilde\rho] &= \lim_{\epsilon\to 0}\frac{\mu^2(\hat\rho+\epsilon\tilde\rho)-\mu^2(\hat\rho)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{\mu^2(\rho_0+(t+\epsilon)\tilde\rho)-\mu^2(\rho_0+t\tilde\rho)}{\epsilon} \\ &= \frac{d\mu^2}{dt}. \end{aligned}$$

Similarly, from the definitions of Q and $\hat L_1/\hat\rho^{1/2}$, it follows that $d_{\hat\rho}Q[\tilde\rho] = \partial Q/\partial t$ and $d_{\hat\rho}(\hat L_1/\hat\rho^{1/2})[\tilde\rho] = (\partial/\partial t)(\hat L_1/\hat\rho^{1/2})$. Define

$$B_2 = \begin{pmatrix} 1/EI & 0 \\ 0 & 1/kAG \end{pmatrix} \tag{48}$$

and

$$\mathscr{F} = -\frac{\hat\rho^{3/4}}{\hat L^2}B_2^{-1}\left\{\left(\frac{d}{dt}\mu^2\right)B_1\hat U + \left(\frac{\partial}{\partial t}Q\right)\hat U + \left(\frac{\partial}{\partial t}\frac{\hat L_1}{\hat\rho^{1/2}}\right)S\hat U_{\hat z}\right\}. \tag{49}$$

Now define

$$\overline{V}(\hat z) \equiv d_{\hat\rho}\hat U[\tilde\rho] = \begin{pmatrix} d_{\hat\rho}\hat\Phi[\tilde\rho] \\ d_{\hat\rho}\hat Y[\tilde\rho] \end{pmatrix} \equiv \begin{pmatrix} \overline{\Phi}(\hat z) \\ \overline{Y}(\hat z) \end{pmatrix}.$$

Then $\overline{V}$ satisfies

$$\begin{gathered} \overline{V}_{\hat z\hat z} + \mu^2 B_1\overline{V} + Q\overline{V} + \frac{\hat L_1}{\hat\rho^{1/2}}S\overline{V}_{\hat z} = \frac{\hat L_1^2}{\hat\rho^{3/4}}B_2\mathscr{F}, \\ \left\{\overline{Y}_{\bar z} - \frac{\hat L_1}{\hat\rho^{1/2}}\overline{\Phi} - \frac{\partial}{\partial t}\left(\frac{\hat L_1}{\hat\rho^{1/2}}\right)\right\}\Bigg|_{\hat z=0,1} = 0, \end{gathered} \tag{50}$$

and

$$\overline{\Phi}_{\bar z}|_{\hat z=0,1} = 0. \tag{51}$$

Let $\bar v$ be defined as

$$\bar v(x) \equiv \begin{pmatrix} \overline{\Psi}(x) \\ \bar y(x) \end{pmatrix} \equiv \hat\rho^{-1/4}(x)\overline{V}(\hat z(x)).$$

From Theorem 2.1, we know that $\bar{v}$ must satisfy the vector differential equation

$$\begin{pmatrix} \frac{\partial}{\partial x} EI \frac{\partial}{\partial x} + \left(p^2 I \hat{\rho} - kAG\right) & kAG \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} kAG & \frac{\partial}{\partial x} kAG \frac{\partial}{\partial x} + p^2 A \hat{\rho} \end{pmatrix} \bar{v}(x) = \mathscr{F}. \quad (52)$$

The boundary conditions (50) and (51) may also be rewritten in terms of $\overline{\Psi}$ and $\bar{y}$. By multiplying (50) through by $\hat{\rho}^{1/4}/\hat{L}_1$ and noting that $\hat{\rho}_x(0) = \hat{\rho}_x(L) = 0$, we find that

$$\left\{ \overline{Y}_{\hat{z}} - \frac{\hat{L}_1}{\hat{\rho}^{1/2}} \overline{\Phi} - \frac{\partial}{\partial t} \left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}} \right) \hat{\Phi} \right\} \bigg|_{\hat{z}=0,1} = 0$$

$$\Leftrightarrow \left\{ \bar{y}_x(x) - \overline{\Psi}(x) - \frac{\hat{\rho}^{1/4}}{\hat{L}_1} \frac{\partial}{\partial t} \left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}} \right) \hat{\Phi}(\hat{z}(x)) \right\} \bigg|_{z=0,L} = 0. \quad (53)$$

Similarly, from (51) we have $\overline{\Phi}_{\hat{z}}|_{\hat{z}=0,1} = 0$

$$\Leftrightarrow \overline{\Psi}_x \big|_{x=0,L} = 0. \quad (54)$$

In the next lemma, we show that if the boundary value problem for $\bar{v}(x)$ is to have a solution when p^2 is an eigenvalue for the boundary value problem (45)–(46), then the right hand side of the differential equation (52) must satisfy a certain orthogonality condition.

LEMMA 5.1. *Let E, I, kG, and A be positive constants, and let $\hat{\rho}$ be a positive function such that $\hat{\rho}_{xx} \in L_\infty(0, L)$. Suppose $\bar{v} = \binom{\overline{\Psi}(x)}{\bar{y}(x)}$ satisfies the non-homogeneous differential equations*

$$EI\overline{\Psi}_{xx} + kAG\left(\bar{y}_x - \overline{\Psi}\right) + p^2 I \hat{\rho} \overline{\Psi} = F_1, \quad (55)$$

and

$$kAG\left(\bar{y}_x - \overline{\Psi}\right)_x + p^2 A \hat{\rho} \bar{y} = F_2, \quad (56)$$

and boundary conditions (53)–(54). *Suppose $\hat{u}(x) = \binom{\hat{\Psi}(x)}{\hat{y}(x)} = \hat{\rho}^{-1/4} \hat{U}(\hat{z}(x))$ solves the homogeneous Timoshenko boundary value problem given in* (45) *and* (46). *Then* $\mathscr{F} = \binom{F_1}{F_2}$ *must satisfy*

$$(\mathscr{F}(x), \hat{u}(x)) = \left(\mathscr{F}, \hat{\rho}^{-1/4}\hat{U}(\hat{z}(x))\right) = \frac{kAG}{\hat{L}_1}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{\Phi}\hat{Y}\bigg|_{\hat{z}=0}^{1}.$$

(In the last equation, $(\cdot,\cdot)$ denotes the standard inner product in $L_2(0, L)$.)

Proof.

$$\begin{aligned}(\mathscr{F}(x), \hat{u}(x)) &= \int_0^L \Big(EI\overline{\Psi}_{xx}\hat{\psi} + kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{\psi} + p^2 I\hat{\rho}\overline{\Psi}\hat{\psi} \\ &\qquad + kAG\big(\bar{y}_x - \overline{\Psi}\big)_x\Big)\hat{y} + p^2 A\hat{\rho}\bar{y}\hat{y}\Big)\,dx \\ &= \int_0^L \Big(EI\overline{\Psi}\hat{\psi}_{xx} + kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{\psi} - kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{y}_x \\ &\qquad + \rho^2 I\hat{\rho}\overline{\Psi}\hat{\psi} + p^2 A\hat{\rho}\bar{y}\hat{y}\Big)\,dx + kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{y}\Big|_{z=0}^{L}\end{aligned}$$

$$\begin{aligned}&= \int_0^L \Big[EI\overline{\Psi}\hat{\psi}_{xx} - kAG\big(\hat{y}_x - \hat{\psi}\big)\big(\bar{y}_x - \overline{\Psi}\big) + p^2 I\hat{\rho}\overline{\Psi}\hat{\psi} + p^2 A\hat{\rho}\bar{y}\hat{y}\Big]\,dx \\ &\quad + kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{y}\Big|_{x=0}^{L} \\ &= \int_0^L \Big\{\Big[EI\hat{\psi}_{xx} + kAG\big(\hat{y}_x - \hat{\psi}\big) + p^2 I\hat{\rho}\hat{\psi}\Big]\overline{\Psi} \\ &\qquad + \Big[kAG\big(\hat{y}_x - \hat{\psi}\big)_x + p^2 A\hat{\rho}\hat{y}\Big]\bar{y}\Big\}\,dx + kAG\big(\bar{y}_x - \overline{\Psi}\big)\hat{y}\Big|_{x=0}^{L} \\ &\qquad = \frac{\hat{\rho}^{1/4}}{\hat{L}_1}kAG\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{\Phi}\hat{\rho}^{-1/4}\hat{Y}\bigg|_{\hat{z}=0}^{1} \\ &\qquad\quad \Rightarrow (\mathscr{F}(x), \hat{u}(x)) = \frac{kAG}{\hat{L}_1}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{\Phi}\hat{Y}\bigg|_{\hat{z}=0}^{1}.\end{aligned}$$

∎

Returning now to (52)–(54), assumptions (i)–(iii) above imply that

$$\bar{v} = \hat{\rho}^{-1/4}\overline{V}(\hat{z}(x)) = \hat{\rho}^{-1/4}\Big(d_{\hat{\rho}}\hat{U}[\tilde{\rho}](\hat{z})\Big) \neq 0$$

is a nontrivial solution to this boundary value problem. Lemma 5.1 implies that

$$\Rightarrow \big(\mathscr{F}(x), \hat{\rho}^{-1/4}\hat{U}(x)\big) = \frac{kAG}{\hat{L}_1}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{\Phi}\hat{Y}\bigg|_{\hat{z}=0}^{1}. \tag{57}$$

Equation (57) holds if and only if

$$\begin{aligned}
&\left(-\frac{\hat{\rho}^{3/4}}{\hat{L}^2}B_2^{-1}\left\{\left(\frac{d}{dt}\mu^2\right)B_1\hat{U} + \left(\frac{\partial}{\partial t}Q\right)\hat{U} + \left(\frac{\partial}{\partial t}\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)S\hat{U}_{\hat{z}}\right\}, \hat{\rho}^{-1/4}\hat{U}\right)\\
&\quad = \frac{kAG}{\hat{L}_1}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{\Phi}\hat{Y}\bigg|_{\hat{z}=0}^{1}\\
&\quad \Leftrightarrow \frac{d}{dt}\mu^2 = \Big\{-\Big(\big(\hat{\rho}^{1/2}/\hat{L}_1\big)B_2^{-1}((\partial/\partial t)Q)\hat{U}, \hat{U}\Big)\\
&\qquad\qquad -\Big(\big(\hat{\rho}^{1/2}/\hat{L}_1\big)\Big((\partial/\partial t)\big(\hat{L}_1/\hat{\rho}^{1/2}\big)\Big)B_2^{-1}S\hat{U}_{\hat{z}}, \hat{U}\Big)\\
&\qquad\qquad -kAG(\partial/\partial t)\big(\hat{L}_1/\hat{\rho}^{1/2}\big)\hat{\Phi}\hat{Y}\Big|_{\hat{z}=0}^{1}\Big\}\\
&\qquad \div\Big(\big(\hat{\rho}^{1/2}/\hat{L}_1\big)B_2^{-1}B_1\hat{U}, \hat{U}\Big).
\end{aligned}$$

The inner products above involve an integration with respect to x. Since $(\hat{\rho}^{1/2}/\hat{L}_1)\,dx = d\hat{z}$, all of the above inner products may be rewritten as integrations with respect to $\hat{z}$. Thus,

$$\begin{aligned}
\frac{d}{dt}\mu^2 = \Big\{&-\int_0^1 \hat{U}^T B_2^{-1}((\partial/\partial t)Q)\hat{U}\,d\hat{z}\\
&-\int_0^1 (\partial/\partial t)\big(\hat{L}_1/\hat{\rho}^{1/2}\big)\hat{U}^T B_2^{-1}S\hat{U}_{\hat{z}}\,d\hat{z}\\
&-kAG(\partial/\partial t)\big(\hat{L}_1/\hat{\rho}^{1/2}\big)\hat{\Phi}\hat{Y}\Big|_{\hat{z}=0}^{1}\Big\}\\
&\div\left(\int_0^1 \hat{U}^T B_2^{-1}B_1\hat{U}\,d\hat{z}\right).
\end{aligned}$$

Next we show that the third term in the numerator above, the boundary term, may be combined with the second term so that all terms in the numerator are integrals. From the definitions of B_2 and S, we find that

$$\int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{U}^T B_2^{-1} S U_{\hat{z}}\, d\hat{z}$$

$$= kAG \int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\left(\hat{Y}_{\hat{z}}\hat{\Phi} - \hat{\Phi}_{\hat{z}}\hat{Y}\right) d\hat{z}$$

$$= 2kAG \int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{Y}_{\hat{z}}\hat{\Phi}\, d\hat{z} - kAG \int_0^1 (\hat{\Phi}\hat{Y})_{\hat{z}} \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) d\hat{z}$$

$$= 2kAG \int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{Y}_{\hat{z}}\hat{\Phi}\, d\hat{z} - kAG\hat{Y}\hat{\Phi}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\Bigg|_0^1$$

$$+ kAG \int_0^1 \hat{\Phi}\hat{Y}\frac{\partial^2}{\partial t\, \partial \hat{z}}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) d\hat{z}$$

$$\Rightarrow -\int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{U}^T B_2^{-1} S \hat{U}_{\hat{z}} d\hat{z} - kAG\hat{Y}\hat{\Phi}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\Bigg|_0^1$$

$$= -2kAG \int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{Y}_{\hat{z}}\hat{\Phi}\, d\hat{z}$$

$$- kAG \int_0^1 \hat{\Phi}\hat{Y}\frac{\partial^2}{\partial t\, \partial \hat{z}}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) d\hat{z}.$$

Therefore,

$$\frac{d}{dt}\mu^2 = \Bigg\{ -\int_0^1 \hat{U}^T B_2^{-1}((\partial/\partial t)Q)\hat{U}\, d\hat{z}$$

$$-2kAG \int_0^1 (\partial/\partial t)\left(\hat{L}_1/\hat{\rho}^{1/2}\right)\hat{Y}_{\hat{z}}\hat{\Phi}\, d\hat{z}$$

$$-kAG \int_0^1 \hat{\Phi}\hat{Y}\left(\partial^2/\partial t\, \partial\hat{z}\right)\left(\hat{L}_1/\hat{\rho}^{1/2}\right) d\hat{z}\Bigg\}$$

$$\div \left(\int_0^1 \hat{U}^T B_2^{-1} B_1 \hat{U}\, d\hat{z}\right). \tag{58}$$

The formula above for $d\mu^2/dt$ holds so long as assumptions (i)–(iii) hold. In the following theorem, conditions are given which guarantee the validity of these assumptions, and hence show when $d\mu^2/dt$ may be calculated using the last formula given above.

THEOREM 5.1. *Suppose* $\rho(x)$ *is a positive function such that* $\rho_x(0) = \rho_x(L) = 0$ *and* $\rho_{xx}(x) \in L_\infty(0, L)$. *Let* $\rho_0 = [(1/L)\int_0^L \rho^{1/2}(x)\,dx]^2$, $\tilde{\rho} \equiv \rho(x) - \rho_0$, *and* $\hat{\rho}(x,t) = \rho_0 + t\tilde{\rho}(x)$. *Let* $\hat{L}_1 = \int_0^L \hat{\rho}^{1/2}(x;t)\,dx$, *and let* $\mu^2(t) = \hat{L}_1^2 \acute{p}^2$ *be an eigenvalue for the transformed boundary value problem* (43)–(44). *Then* $d\mu^2/dt$ *satisfies* (7), *where* $\hat{U}$ *is an eigenfunction of* (43)–(44) *corresponding to the eigenvalue* μ^2, *and* Q, B_1, *and* B_2 *are the* 2×2 *matrices defined in* (41) *and in* (48).

Proof. The proof follows from the discussion preceding this theorem, provided we show assumptions (i)–(iii) are valid. We observe that $\hat{u}(x) = \hat{\rho}^{-1/4}(x)\hat{U}(\hat{z}(x))$, where $\hat{u}(x)$ satisfies (45) and (46). The differential equation for the vector $\hat{u}(x)$ may be written as a system of four first order linear scalar equations in which the parameter t appears linearly. This implies that for each x, $\hat{u}$, $\hat{u}_x$, and $\hat{u}_{xx}$ must be analytic in t. See Coddington and Levinson [4, p. 37]. This in turn implies that for each $\hat{z}$, $\hat{U}(\hat{z})$, $\hat{U}_{\hat{z}}(\hat{z})$, and $\hat{U}_{\hat{z}\hat{z}}(\hat{z})$ must also be analytic in t when $t \in [0,1]$. When $\hat{z} = 0$ and t is any value between 0 and 1, both components of $\hat{U}(0)$ cannot simultaneously be zero. (If they were both zero, boundary conditions at $\hat{z} = 0$ would imply $\hat{U}(\hat{z})$ must be identically zero.) From the scalar differential equations for the components of $\hat{U}(\hat{z})$, we conclude that μ^2 must be analytic in t. We have shown already that $d_{\hat{\rho}}\mu^2[\tilde{\rho}] = d\mu^2/dt$; hence, assumption (i), that $d_{\hat{\rho}}\mu^2[\tilde{\rho}]$ exists, is valid.

Next, we observe that $d_{\hat{\rho}}\hat{U}[\tilde{\rho}] = \frac{\partial}{\partial t}\hat{U}$, $d_{\hat{\rho}}\hat{U}_{\hat{z}}[\tilde{\rho}] = \frac{\partial}{\partial t}\hat{U}_{\hat{z}}$, and $d_{\hat{\rho}}\hat{U}_{\hat{z}\hat{z}}[\tilde{\rho}] = \frac{\partial}{\partial t}\hat{U}_{\hat{z}\hat{z}}$. From this we conclude that $d_{\hat{\rho}}\hat{U}[\tilde{\rho}]$, $d_{\hat{\rho}}\hat{U}_{\hat{z}}[\tilde{\rho}]$, and $d_{\hat{\rho}}\hat{U}_{\hat{z}\hat{z}}[\tilde{\rho}]$ all exist, since $\hat{U}$, $\hat{U}_{\hat{z}}$, and $\hat{U}_{\hat{z}\hat{z}}$ are all analytic in t. This proves assumption (ii).

To demonstrate that assumption (iii) holds under the stated hypotheses, observe that the components of $\hat{U}$ satisfy the integral equations (20) and (21) when ρ is taken to be $\hat{\rho}$. When $\rho_{xx}(x)$ is continuous, it follows from these integral equations that $\hat{U}_{\hat{z}\hat{z}t}$ and $\hat{U}_{t\hat{z}\hat{z}}$ are continuous functions of $\hat{z}$ and t. This implies that when $\rho(x) \in C_2[0, L]$, $\hat{U}_{\hat{z}\hat{z}t} = \hat{U}_{t\hat{z}\hat{z}}$. Furthermore, from the integral equations, it follows that $\hat{U}_{\hat{z}\hat{z}t}$ and $\hat{U}_{t\hat{z}\hat{z}}$ are continuous in ρ with respect to the standard Sobelev norm of order 2. They are continuous maps which take $\rho(x) \in H_2(0, L)$ to an element of $L_2(0,1)$. Since $C_2[0, L]$ is dense in $H_2(0, L)$, it follows that $\hat{U}_{\hat{z}\hat{z}t} = \hat{U}_{t\hat{z}\hat{z}}$ when $\rho(x) \in H_2(0, L)$. Since $H_2(0, L)$ contains the set of all ρ such that $\rho_{xx} \in L_\infty(0, L)$, assumption (iii) must hold when $\rho_{xx} \in L_\infty[0, L]$, as desired. ∎

In the next three lemmas, our goal is to determine the constants c and d which appear in the integral equations (20) and (21). Theorem 4.1 shows that if p^2 is a large enough eigenvalue, then p must lie near a root of $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$. We will show that provided p is not near more than one root of $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$, then p must be a simple eigenvalue, and the vector $(c, d)^T$ must be a multiple of either

$(1, O(1/p))^T$ or $(O(1/p), 1)^T$, depending upon whether p is near a zero of $\sin(\hat{L}_1 p/\sqrt{kG})$ or of $\sin(\hat{L}_1 p/\sqrt{E})$. This result will be used to calculate an estimate of $d\mu^2/dt$.

In the next lemma, we show that if $\hat{p}$ is large enough and satisfies either (59) or (60) below, then the square root of the nearest eigenvalue to $\hat{p}$ is well separated from square roots of other eigenvalues.

LEMMA 5.2. *Let $e \in (0,1)$ and suppose $\hat{p}$ is a root of the function $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$ such that either*

$$\sin\left(\hat{L}_1 \hat{p}/\sqrt{kG}\right) = 0 \qquad \textit{and} \qquad \left|\sin\left(\hat{L}_1 \hat{p}/\sqrt{E}\right)\right| > e \tag{59}$$

or

$$\sin\left(\hat{L}_1 \hat{p}/\sqrt{E}\right) = 0 \qquad \textit{and} \qquad \left|\sin\left(\hat{L}_1 \hat{p}/\sqrt{kG}\right)\right| > e. \tag{60}$$

Then there exists an $M > 0$ such that when $\hat{p} > M$, exactly one eigenvalue of the free-free Timoshenko beam with density $\hat{\rho}$, say $\acute{p}^2$, is such that its square root $\acute{p}$ lies in the interval $[\hat{p} - e\sqrt{\underline{c}}/2\hat{L}_1, \hat{p} + e\sqrt{\underline{c}}/2\hat{L}_1]$. Furthermore, $|\acute{p} - \hat{p}| < O(1/\acute{p})$.

Proof. It is not difficult to show that when (59) or (60) hold, and when $\tilde{p}$ is a root of $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$ as near to $\hat{p}$ as is any other root of this function, then $|\hat{p} - \tilde{p}| > e\sqrt{\underline{c}}/\hat{L}_1$, where $\underline{c} = \min E, kG$. Let Γ be defined as it is in the hypothesis of Theorem 4.3. Let M be chosen large enough so that $\hat{p} > M$ implies that hypothesis (37) of Theorem 4.3 is satisfied and that $|\Gamma| < e\sqrt{\underline{c}}/[2\hat{L}_1]$. Theorem 4.1 shows that if there is a root $\acute{p}$ of the frequency equation such that $\acute{p} \in [\hat{p} - e\sqrt{\underline{c}}/2\hat{L}_1, \hat{p} + e\sqrt{\underline{c}}/2\hat{L}_1]$, then $\acute{p}$ must be contained in the narrower interval $(\hat{p} - \Gamma, \hat{p} + \Gamma)$. On the other hand, Theorem 4.3 shows that there is exactly one zero of the frequency function in the interval $(\hat{p} - \Gamma, \hat{p} + \Gamma)$. Together, Theorem 4.1 and Theorem 4.3 imply that there is exactly one root of the frequency function in the interval $[\hat{p} - e\sqrt{\underline{c}}/2\hat{L}_1, \hat{p} + e\sqrt{\underline{c}}/2\hat{L}_1]$. ∎

LEMMA 5.3. *Let $e \in (0,1)$, and suppose $\hat{p}$ is a root of $\sin(\hat{L}_1 p/\sqrt{kG} \cdot \sin(\hat{L}_1 p/\sqrt{E})$ such that either (59) or (60) holds. For $\hat{p}$ large enough, let $\acute{p}^2$ be the unique eigenvalue of the free-free beam with density $\hat{\rho}$ whose square root $\acute{p}$ is contained in the interval $[\hat{p} - e\sqrt{\underline{c}}/2\hat{L}_1, \hat{p} + e\sqrt{\underline{c}}/2\hat{L}_1]$. If (59) holds, then for the eigenvalue $\acute{p}^2$, the vector $(c, d)^T$ (where c and d are the constants appearing in (20) and (21)) must satisfy*

$$\begin{pmatrix} c \\ d \end{pmatrix} = constant \begin{pmatrix} O(1/\acute{p}) \\ 1 \end{pmatrix}.$$

if (60) *holds*,

$$\begin{pmatrix} c \\ d \end{pmatrix} = constant \begin{pmatrix} 1 \\ O(1/\acute{p}) \end{pmatrix}.$$

In either case, whether (59) *holds or* (60) *holds,* $\acute{p}^2$ *must be a simple eigenvalue.*

Proof. $\acute{p}^2$ is an eigenvalue if and only if nontrivial solutions exist to a linear system of the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} O(1/\acute{p}) & -\sin\left(\dfrac{\acute{p}\hat{L}_1}{\sqrt{kG}}\right) - O(1/\acute{p}) \\ -\sin\left(\dfrac{\acute{p}\hat{L}_1}{\sqrt{E}}\right) - O(1/\acute{p}) & -O(1/\acute{p}) \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \quad (61)$$

(see Theorem 2.15). The conclusion is immediate. ∎

Our next goal is to use formula (58) in Theorem 5.1 to calculate an estimate for $d\mu^2/dt$. We make an estimate $d\mu^2/dt$ in the case where $\acute{p}$ lies near a root $\hat{p}$ of $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$ which for an arbitrary but fixed $e \in (0,1)$ satisfies either (59) or (60). A $\acute{p}$ which meets this criterion gives rise to an eigenvalue which we will refer to as being "well-separated" from its neighboring eigenvalues. The estimate of $d\mu^2/dt$ is used to calculate asymptotic formulas for these eigenvalues.

In the next lemma, we calculate estimates for one of the terms appearing in the numerator of the right hand side of (58).

LEMMA 5.4. *Let* $\hat{U}$ *and* $\mu^2 = \hat{L}_1^2 \acute{p}^2$ *be an eigenfunction and corresponding eigenvalue for the transformed boundary value problem* (43)–(44). *Then* $\acute{p}^2$ *is an eigenvalue for the free-free Timoshenko beam with density* $\hat{\rho}$. *Let* e *be a fixed but arbitrary number in* $(0,1)$, *and let* $\hat{p}$ *be a root of* $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$ *as near to* $\acute{p}$ *as is any other root of this function. If* $\hat{p}$ *satisfies either* (59) *or* (60), *then*

$$-2kAG\int_0^1 \frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{Y}_s\hat{\Phi}\, ds$$

$$= \left[\left(\frac{c}{\alpha}\right)^2 - d^2\frac{A}{I}\right]\left[\frac{kAG}{\dfrac{E}{kG} - 1}\right]\int_0^1 \left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) ds + O(1/\acute{p}).$$

Proof. Theorem 2.2 shows that $\acute{p}^2$ is an eigenvalue for the free-free Timoshenko beam when $L_1^2\acute{p}^2$ is an eigenvalue for the transformed problem.

Let $\hat{\mu}_E = \hat{L}_1\acute{p}/\sqrt{E}$ and $\hat{\mu}_{kG} = \hat{L}_1\acute{p}/\sqrt{kG}$. Then Theorem 3.1 shows that

$$\begin{aligned}\hat{Y}_{\hat{z}} &= c\cos(\hat{\mu}_{kG}\hat{z}) - d\hat{\mu}_{kG}\sin(\hat{\mu}_{kG}\hat{z})\\ &\quad + \int_0^{\hat{z}}\cos[\hat{\mu}_{kG}(\hat{z}-s)]\\ &\quad\quad \times\left\{-\hat{\rho}_3(s)\hat{Y}(s) - \hat{\rho}_4(s)\hat{\Phi}(s) + \frac{\hat{L}_1}{\hat{\rho}^{1/2}(s)}\hat{\Phi}_s(s)\right\}ds\end{aligned}$$

and

$$\begin{aligned}\hat{\Phi} &= \frac{c\cos(\hat{\mu}_E\hat{z})}{\alpha} + \int_0^{\hat{z}}\frac{\sin[\hat{\mu}_E(\hat{z}-s)]}{\hat{\mu}_E}\\ &\quad\cdot\left\{-\hat{\rho}_3(s)\hat{\Phi}(s) + \frac{\hat{L}_1^2\gamma}{\hat{\rho}(s)}\hat{\Phi}(s)\right.\\ &\quad\quad\left. + \gamma\hat{\rho}_4(s)\hat{Y}(s) - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}(s)}\hat{Y}(s)\right\}ds.\end{aligned}$$

Let $f(\hat{z}) \equiv 2kAG(\partial/\partial t)(\hat{L}_1/\hat{\rho}^{1/2})$. Then

$$\begin{aligned}&\int_0^1 f(\hat{z})\hat{Y}_{\hat{z}}(\hat{z})\hat{\Phi}(\hat{z})\,d\hat{z}\\ &\quad = \int_0^1 f\left[\frac{c^2}{\alpha}\cos(\hat{\mu}_{kG}\hat{z})\cos(\hat{\mu}_E\hat{z}) - \frac{cd}{\alpha}\hat{\mu}_{kG}\sin(\hat{\mu}_{kG}\hat{z})\cos(\hat{\mu}_E\hat{z})\right]d\hat{z}\\ &\quad\quad - d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1 f(\hat{z})\sin(\hat{\mu}_{kG}\hat{z})\\ &\quad\quad\quad \times\int_0^{\hat{z}}\sin[\hat{\mu}_E(\hat{z}-s)]\left\{-\hat{\rho}_3\hat{\Phi} + \frac{\hat{L}_1^2\gamma}{\hat{\rho}}\hat{\Phi} + \gamma\hat{\rho}_4\hat{Y} - \frac{\hat{L}_1\gamma}{\hat{\rho}^{1/2}}\hat{Y}_s\right\}ds\,d\hat{z}\\ &\quad\quad\quad \int_0^1 f(\hat{z})\frac{c}{\alpha}\cos(\hat{\mu}_E\hat{z})\int_0^{\hat{z}}\cos[\hat{\mu}_{kG}(\hat{z}-s)]\\ &\quad\quad\quad\quad \times\left\{-\hat{\rho}_3\hat{Y} - \hat{\rho}_4\hat{\Phi} + \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\hat{\Phi}_s\right\}ds\,d\hat{z} + O\left(\frac{1}{\acute{p}}\right).\end{aligned}$$

By Lemma 5.3, when $\acute{p}$ satisfies (59) or (60), then $\frac{cd}{\alpha}\hat{\mu}_{kG}$ and $\frac{c}{\alpha}$ must both be $O(1)$ or smaller. Since $\hat{\mu}_E \neq \hat{\mu}_{kG}$ by assumption (otherwise neither (59) nor (60) could hold), the first integral on the right of the above equation must be $O(1/\acute{p})$.

Let

$$I_1 = -d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1 f(\hat{z})\sin(\hat{\mu}_{kG}\hat{z})$$
$$\cdot\left[\int_0^{\hat{z}}\sin[\hat{\mu}_E(\hat{z}-s)]\left\{-\hat{\rho}_3\hat{\Phi} + \frac{\hat{L}_1^2\gamma}{\hat{\rho}}\hat{\Phi} + \gamma\hat{\rho}_4\hat{Y} - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}}\hat{Y}_s\right\}ds\right]d\hat{z}.$$

By switching the order of integration, we find that

$$I_1 = -d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1\int_s^1 f(\hat{z})\sin(\hat{\mu}_{kG}\hat{z})\sin[\hat{\mu}_E(\hat{z}-s)]$$
$$\cdot\left\{-\hat{\rho}_3(s)\hat{\Phi}(s) + \frac{\hat{L}_1^2\gamma}{\hat{\rho}(s)}\hat{\Phi}(s) + \gamma\hat{\rho}_4(s)\hat{Y}(s) - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}(s)}\hat{Y}_s(s)\right\}d\hat{z}\,ds$$
$$= -d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1\left\{-\hat{\rho}_3 + \frac{\hat{L}_1^2\gamma}{\hat{\rho}}\hat{\Phi} + \gamma\hat{\rho}_4\hat{Y} - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}}\hat{Y}_s\right\}$$
$$\cdot\left[\int_s^1\frac{f(\hat{z})}{2}\{\cos[(\hat{\mu}_{kG}-\hat{\mu}_E)\hat{z}+\hat{\mu}_E s]\right.$$
$$\left.-\cos[(\hat{\mu}_{kG}+\hat{\mu}_E)\hat{z}-\hat{\mu}_E s]\}\,d\hat{z}\right]ds.$$

Now $f'' \in L_\infty(0,1)$ because $\hat{\rho}_{xx} \in L_\infty(0,L)$. Therefore we may integrate the square bracketed term above by parts with respect to $\hat{z}$ and apply Lemmas 3.2, 3.3, 3.4, and 3.5 to show that

$$I_1 = -d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1\left\{-\hat{\rho}_3\hat{\Phi} + \frac{\hat{L}_1^2\gamma}{\hat{\rho}}\hat{\Phi} + \gamma\hat{\rho}_4\hat{Y} - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}}\hat{Y}_s\right\}$$
$$\cdot\frac{f}{2}\left[\frac{1}{\hat{\mu}_{kG}+\hat{\mu}_E} - \frac{1}{\hat{\mu}_{kG}-\hat{\mu}_E}\right]\sin(\hat{\mu}_{kG}s)\,ds + O(1/\acute{p})$$
$$= -d\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1\frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}}\frac{f}{2}\frac{2\hat{\mu}_E}{\hat{\mu}_{kG}^2-\hat{\mu}_E^2}\sin(\hat{\mu}_{kG}s)\hat{Y}_s ds + O(1/\acute{p}).$$

From the integral equation for $\hat{Y}_{\hat{z}}$, it follows that $\hat{Y}_{\hat{z}} = -d\hat{\mu}_{kG}\sin(\hat{\mu}_{kG}\hat{z}) + O(1)$. Substituting this expression for $\hat{Y}_{\hat{z}}$ into the last expression for I_1, we find that

$$\begin{aligned} I_1 &= d^2\frac{\hat{\mu}_{kG}}{\hat{\mu}_E}\int_0^1 \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}} f \frac{\hat{\mu}_{kG}\hat{\mu}_E}{\hat{\mu}_{kG}^2 - \hat{\mu}_E^2}\sin^2(\hat{\mu}_{kG}s)\,ds + O(1/\acute{p}) \\ &= d^2 kAG\frac{A}{I}\left(\frac{1}{\dfrac{E}{kG} - 1}\right)\int_0^1 \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) ds + O(1/\acute{p}). \end{aligned}$$

Using a similar argument as was used to derive the estimate of I_1, it is possible to show that

$$\begin{aligned} I_2 &= \int_0^1 f(\hat{z})\frac{c}{\alpha}\cos(\hat{\mu}_E\hat{z}) \\ &\quad\times\int_0^{\hat{z}}\cos[\hat{\mu}_{kG}(\hat{z} - s)]\left\{-\hat{\rho}_3\hat{Y} - \hat{\rho}_4\hat{\Phi} + \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\hat{\Phi}_s\right\} ds\,d\hat{z} \\ &= -\left[\frac{c}{\alpha}\right]^2 kAG\left(\frac{1}{E/kG - 1}\right)\int_0^1 \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) ds + O(1/\acute{p}). \end{aligned}$$

Since

$$\int_0^1 2kAG\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right)\hat{Y}_s\hat{\Phi}\,ds = I_1 + I_2 + O(1/\acute{p}),$$

the lemma follows. ∎

In the next theorem, we make estimates for transformed eigenfunctions $\hat{U}(\hat{z})$ associated with well separated eigenvalues for the untransformed Timoshenko beam with density $\hat{\rho}$. We will use these estimates for $\hat{U}$ to calculate an estimate for $d\mu^2/dt$.

LEMMA 5.5. *Suppose $\acute{p}$ is an eigenvalue for the free-free Timoshenko beam with density $\hat{\rho} = \rho_0 + t\tilde{\rho}(x)$, where $\rho_0 = ([1/L]\int_0^L \rho^{1/2}(x)\,dx)^2$ and $\tilde{\rho}(x) = \rho(x) - \rho_0$. Suppose $\rho(x)$ is a positive function on $[0, L]$, that $\rho_{xx}(x) \in L_\infty(0, L)$, and that $\rho_x(0) = \rho_x(L) = 0$. Assume also that E, I, kG, and A are positive constants. Let $\hat{p}$ be a root of $\sin(\hat{L}_1 p/\sqrt{kG})\cdot\sin(\hat{L}_1 p/\sqrt{E})$ as near to $\acute{p}$ as is any other root of this function. Let e be an arbitrary but fixed constant between 0 and 1. If (59) holds, then for some integer n, $\hat{p} = n\pi\sqrt{kG}/\hat{L}_1$, and $\mu^2 = \hat{L}_1^2\acute{p}^2$, $\hat{U}(\hat{z}) = (\cos(n\pi\hat{z}) + O(1/n), O(1/n))^T$ is an eigenvalue-eigenfunction pair for (43)–(44). Similarly, if $\hat{p}$ satisfies (60), then* *for some integer* n, $\hat{p} = n\pi/\sqrt{E}/\hat{L}_1$ *and an eigenvalue-eigenfunction pair for* (43)–(44) *is* $\mu^2 = \hat{L}_1^2 \acute{p}^2$, $\hat{U}(\hat{z}) = (O(1/n), \cos(n\pi\hat{z}) + O(1/n))^T$.

Proof. If $\hat{\mu}_E = \hat{L}_1 \acute{p}/\sqrt{E}$ and $\hat{\mu}_{kG} = \hat{L}_1 \acute{p}/\sqrt{kG}$, then Theorem 3.1 shows that for a nontrivial choice of the constants c and d and for $\hat{\alpha} = \hat{L}_1/\hat{\rho}^{1/2}(0)$,

$$\hat{\Phi} = \frac{c\cos(\hat{\mu}_E \hat{z})}{\hat{\alpha}}$$

$$\int_0^{\hat{z}} \frac{\sin[\hat{\mu}_E(\hat{z}-s)]}{\hat{\mu}_E}\left\{-\hat{\rho}_3\hat{\Phi} + \frac{\hat{L}_1^2\gamma}{\hat{\rho}} + \gamma\hat{\rho}_4\hat{Y} - \frac{\gamma\hat{L}_1}{\hat{\rho}^{1/2}}\hat{Y}_s\right\} ds \qquad (62)$$

and

$$\hat{Y} = \frac{c\sin(\hat{\mu}_{kG}\hat{z})}{\hat{\mu}_{kG}} + d\cos(\hat{\mu}_{kG}\hat{z})$$

$$\int_0^{\hat{z}} \frac{\sin[\hat{\mu}_{kG}(\hat{z}-s)]}{\hat{\mu}_{kG}}\left\{-\hat{\rho}_3\hat{Y} - \hat{\rho}_4\hat{\Phi} + \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\hat{\Phi}_s\right\} ds. \qquad (63)$$

For fixed c and d, Lemmas 3.3, 3.4, and 3.5 show that all terms in (62) and (63) under the integral sign are $O(1/\acute{p})$. If (59) holds, then by Lemma 5.3 and Theorem 4.1 we find that $\acute{p} = \hat{p} + O(1/\hat{p}) = n\pi\hat{L}_1/\sqrt{kG} + O(1/n)$. This implies that $\hat{U}(\hat{z}) = (O(1/n), \cos(n\pi\hat{z}) + O(1/n))^T$. Similarly, when (60) holds, $\hat{U}(\hat{z}) = (\cos(n\pi\hat{z}) + O(1/n), O(1/n))^T$. ∎

The next lemma gives an estimate of $d\mu^2/dt$.

LEMMA 5.6. *Suppose* e *is an arbitrary but fixed real number between* 0 *and* 1. *Suppose* $\acute{p}^2$ *is an eigenvalue for the free-free Timoshenko beam with density* $\hat{\rho}(x;t)$ *and that* $\hat{p}$ *is a root of the function* $\sin(\hat{L}_1 p/\sqrt{kG}) \cdot \sin(\hat{L}_1 p/\sqrt{E})$ *nearest* $\acute{p}$. *Let* $\mu^2 = \hat{L}_1^2\acute{p}^2$. *If* (59) *holds, then* $\hat{p} = n\pi\sqrt{kG}/\hat{L}_1$ *for some integer* n, *and*

$$\frac{d\mu^2}{dt} = -kG\int_0^1 \{(\hat{\rho}_3)_t[1 + \cos(2n\pi z)]\}\, dz$$

$$- \frac{2kAG}{I(E/kG - 1)}\int_0^1 \frac{\hat{L}_1}{\hat{\rho}^{1/2}}\frac{\partial}{\partial t}\left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}}\right) dz + O(1/n). \quad (64)$$

On the other hand, if (60) *holds, then* $\hat{p} = n\pi\sqrt{E}/\hat{L}_1$ *for some integer* n, *and*

$$\frac{d\mu^2}{dt} = E\int_0^1 \left\{ \left[\frac{\hat{L}_1^2\gamma}{\hat{\rho}} - \hat{\rho}_3 \right]_t - \cos(2n\pi z)(\hat{\rho}_3)_t \right\} dz$$

$$+ \frac{2kAG}{I(E/kG - 1)} \int_0^1 \frac{\hat{L}_1}{\hat{\rho}^{1/2}} \frac{\partial}{\partial t} \left(\frac{\hat{L}_1}{\hat{\rho}^{1/2}} \right) dz + O(1/n). \quad (65)$$

Proof. The proof follows from Theorem 5.1 and Lemmas 5.4 and 5.5. ∎

Finally, for eigenvalues which are well separated from neighboring eigenvalues, we prove the following asymptotic formulas.

THEOREM 5.2. *Let* A, I, E, *and* kG *be positive constants. Suppose that* $\rho(x) > 0$ *for all* $x \in [0, L]$, *that* $\rho_x(0) = \rho_x(L) = 0$, *and that* $\rho_{xx} \in L_\infty[0, L]$. *Let* $\rho_0 = ([1/L]\int_0^L \rho^{1/2}(x)\,dx)^2$, *and define*

$$\mathscr{E}(p) = \sin\left(\frac{pL\rho_0^{1/2}}{\sqrt{E}} \right) \sin\left(\frac{pL\rho_0^{1/2}}{\sqrt{kG}} \right).$$

Let e *be a fixed but arbitrary real number between* 0 *and* 1, *and let*

$$p_{n_i} \equiv \frac{n_i\pi\sqrt{E}}{L\rho_0^{1/2}}, \qquad i = 1, 2, \ldots,$$

be a sequence of roots of $\mathscr{E}(p)$ *such that*

$$\left| \sin\left(\frac{p_{n_i} L\rho_0^{1/2}}{\sqrt{kG}} \right) \right| > e.$$

Suppose each p_{n_i} *is large enough that exactly one simple eigenvalue* $\check{p}^2$ *of the Timoshenko beam satisfies*

$$\check{p} \in \left[p_{n_i} - \frac{\sqrt{\underline{c}}}{2L\rho_0^{1/2}}e,\ p_{n_i} + \frac{\sqrt{\underline{c}}}{2L\rho_0^{1/2}}e \right],$$

where $\underline{c} = \min\{E, kG\}$. *Then*

$$\check{p}^2 = \frac{n_i^2\pi^2 E}{L^2\rho_0} - \frac{E}{L^2\rho_0}\int_0^1 \rho_3(x)\cos(2n_i\pi z)\,dz + C_E + O(1/n_i), \quad (66)$$

where $x(z)$ is the inverse of the function $z(x) = [1/L_1]\int_0^x \rho^{1/2}(s)\,ds$ and

$$C_E = \frac{A}{I}\frac{kG}{\rho_0}\left[\frac{E}{E - kG}\right]\int_0^1 \left(\frac{\rho_0}{\rho(x)} - 1\right) dz - \frac{E}{L^2\rho_0}\int_0^1 \rho_3(x)\,dz + \frac{A}{I}\frac{kG}{\rho_0}\left(1 - \frac{kG + E}{2(E - kG)}\right). \tag{67}$$

Let

$$p_{m_i} \equiv \frac{m_i \pi \sqrt{kG}}{L\rho_0^{1/2}}, \qquad i = 1, 2, \dots,$$

be a sequence of roots of $\mathscr{E}(p)$ such that

$$\left|\sin\left(\frac{p_{m_i} L \rho_0^{1/2}}{\sqrt{E}}\right)\right| > e.$$

Suppose each p_{m_i} is large enough that there is exactly one simple eigenvalue $\check{p}^2$ of the Timoshenko beam satisfying

$$\check{p} \in \left[p_{m_i} - \frac{\sqrt{\underline{c}}}{2L\rho_0^{1/2}}e,\ p_{m_i} + \frac{\sqrt{\underline{c}}}{2L\rho_0^{1/2}}e\right].$$

In this case,

$$\check{p}^2 = \frac{m_i^2 \pi^2 kG}{L^2\rho_0} - \frac{kG}{L^2\rho_0}\int_0^1 \rho_3(x)\cos(2m_i\pi z)\,dz + C_{kG} + O(1/m_i), \tag{68}$$

where

$$C_{kG} = -\frac{A}{I}\frac{kG}{\rho_0}\left[\frac{kG}{E - kG}\right]\int_0^1 \left(\frac{\rho_0}{\rho(x)} - 1\right) dz - \frac{kG}{L^2\rho_0}\int_0^1 \rho_3(x)\,dz + \frac{A}{I}\frac{kG}{\rho_0}\left(1 + \frac{kG + E}{2(E - kG)}\right). \tag{69}$$

Proof. To prove formula (66) let $\mu^2 = \hat{L}_1^2 \acute{p}^2$ where $\acute{p}^2$ is an eigenvalue for a beam with density $\hat{\rho}(x; t)$. Define $\check{p}^2 = \acute{p}^2|_{t=1}$ and $\bar{p}^2 = \acute{p}^2|_{t=0}$. Because $\hat{L}_1^2|_{t=1} = \hat{L}_1^2|_{t=0} = L^2\rho_0 = L_1^2$, as t changes from 0 to 1, $\mu^2/[L^2\rho_0]$ will change continuously in t from $\bar{p}^2$, an eigenvalue for a uniform beam with constant density ρ_0, to $\check{\rho}^2$, an eigenvalue for a beam with density $\rho(x)$. Furthermore, Lemma 5.6 shows that $d\mu^2/dt$ satisfies (65). This implies that

$$L^2\rho_0\left(\check{p}^2 - \bar{p}^2\right) = E\int_0^1 \left\{ L_1^2 \gamma \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) - \rho_3 - \cos(2n\pi z)\,\rho_3 \right\} dz$$
$$+ \frac{L^2 kAG}{I(E/kG - 1)} \int_0^1 \left[\frac{\rho_0}{\rho} - 1 \right] dz + O(1/n_i), \quad (70)$$

where $L^2\rho_0\,\check{p}^2 = \mu^2|_{t=1}$ is an eigenvalue for the transformed problem with density $\rho(x)$ and $L^2\rho_0\,\bar{p}^2 = \mu^2|_{t=0}$ is an eigenvalue for the transformed problem with constant density ρ_0. Dividing through by $L^2\rho_0 = L_1^2$, we find that

$$\check{p}^2 - \bar{p}^2 = E\int_0^1 \gamma\left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) dz - \frac{E}{L^2\rho_0} \int_0^1 \rho_3 dz$$
$$+ \frac{kAG}{I(E/kG - 1)} \int_0^1 \left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) dz$$
$$- \frac{E}{L^2\rho_0} \int_0^1 \cos(2n_i\pi z)\,\rho_3 dz + O\left(\frac{1}{n_i} \right). \quad (71)$$

From Geist and McLaughlin [10], it follows that

$$\bar{p}^2 = \frac{n_i^2\pi^2 E}{L^2\rho_0} + \left(1 - \frac{1}{2}\,\frac{kG + E}{kG - E} \right) \frac{A}{I}\,\frac{kG}{\rho_0} + O\left(\frac{1}{n_i} \right).$$

Formula (66) follows from the last equation and (71).

A similar argument proves formulas (68) and (69). For eigenvalues that are close to the $p_{m_i}s$, $d\mu^2/dt$ satisfies (64) (instead of (65)), and the formula for $\bar{p}^2$ becomes (see [10])

$$\bar{p}^2 = \frac{m_i^2\pi^2 kG}{L^2\rho_0} + \left(1 + \frac{1}{2}\,\frac{kG + E}{kG - E} \right) \frac{A}{I}\,\frac{kG}{\rho_0} + O\left(\frac{1}{m_i} \right).$$

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REFERENCES

1. G. Bachman and L. Narici, "Functional Analysis," Academic Press, New York, 1966.
2. G. Borg, Eine umkehrung der Sturm–Liouvilleschen eigenwertaufgabe, *Acta Math*. **78** (1946), 1–96.

3. W. E. Boyce and R. C. Diprima, "Elementary Differential Equations and Boundary Value Problems," 5th ed., Wiley, New York, 1992.
4. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw–Hill, New York, 1955.
5. C. F. Coleman and J. R. McLaughlin, Solution of the inverse spectral problem for an impedance with integrable derivative, parts I, II, *Comm. Pure Appl. Math.* **46** (1993), 145–212.
6. C. T. Fulton and S. A. Pruess, Eigenvalue and eigenfunction asymptotics for regular Sturm–Liouville problems, *J. Math. Anal. Appl.* **188** (1994), 297–340.
7. C. T. Fulton and S. A. Pruess, Erratum, *J. Math. Anal. Appl.* **189** (1995).
8. B. Geist, "The Asymptotic Expansion of the Eigenvalues of the Timoshenko Beam," Ph.D. dissertation, Rensselaer Polytechnic Institute, Troy, NY, 1994.
9. B. Geist and J. R. McLaughlin, Double eigenvalues for the uniform Timoshenko beam, *Appl. Math. Lett.* **10**, No. 3 (1997), 129–134.
10. B. Geist and J. R. McLaughlin, Eigenvalue formulas for the uniform Timoshenko beam: The free-free problem, *Electron. Res. Announc. Amer. Math. Soc.* **4** (1998).
11. O. H. Hald, The inverse Sturm–Liouville problem with symmetric potentials, *Acta Math.* **141** (1978), 263–291.
12. O. H. Hald and J. R. McLaughlin, Solutions of inverse nodal problems, *Inverse Problems* **5** (1989), 307–347.
13. T. C. Huang, The effect of rotatory inertia and of shear deformation on the frequency and normal mode equations of uniform beams with simple end conditions, *J. Appl. Mech.* **28** (1961), 579–584.
14. E. L. Isaacson and E. Trubowitz, The inverse Sturm–Liouville problem, I, *Comm. Pure Appl. Math.* **36** (1983), 767–783.
15. E. T. Kruszewski, Effect of transverse shear and rotary inertia on the natural frequency of a uniform beam, *in* "National Advisory Committee for Aeronautics," Technical Note, No. 1909, July 1949.
16. B. Noble and J. W. Daniel, "Applied Linear Algebra," 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1977.
17. J. Poschel and E. Trubowitz, "Inverse Spectral Theory," Academic Press, New York, 1987.
18. W. Rundell and P. Sacks, Reconstruction techniques for classical inverse Sturm–Liouville problems, *Math. Comp.* **58**, No. 197 (1992), 161–183.
19. C. R. Steele, Application of the WKB method in solid mechanics, *Mech. Today* **3** (1976), 243–295.
20. W. T. Thomson, "Vibration Theory and Applications," Prentice-Hall, Englewood Cliffs, NJ, 1965.
21. S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philos. Mag.* **41** (1921), 744–746.
22. S. Timoshenko, On the transverse vibrations of bars of uniform cross-section, *Philos. Mag.* **43** (1922), 125–131.
23. S. Timoshenko, "Strength of Materials: Elementary Theory and Problems," 2nd ed., Van Nostrand, New York, 1940.
24. R. W. Trail-Nash and A. R. Collar, The effects of shear flexibility and rotatory inertia on the bending vibrations of beams, *Quart. J. Mech. Appl. Math.* **6**, No. 2 (1953), 186–222.